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CDMA Capacity

Santosh S. Venkatesh
Department of Electrical and Systems Engineering
University of Pennsylvania
Philadelphia, PA 19104
e-mail:
venkatesh@ee.upenn.edu

Abstract — **How many users can be supported in a direct sequence CDMA system? It is shown that under some conditions, a threshold function (or phase transition) is manifested in the number of users that can be supported. This phenomenon is essentially combinatorial in nature and may be explicated by random graph methods.**

I. Introduction

Code-division multiple access schemes allow a multiplexing of resources among several users sharing a multiaccess channel. In such settings, each user modulates a preassigned (pseudo-) random signature waveform and transmits information independently of, and possibly concurrently with, other users. The individual signature waveforms are known to the receiver who can resort to a variety of schemes to attempt to recover the data transmitted by each user. The performance measure of interest is the probability of error for each user.

While sundry factors impact receiver performance including variable and random user transmission delays occasioned by asynchronous transmission, uneven power distribution (the near-far problem), the complexity of the receiver's algorithm, and additive channel noise, a fundamental limitation is imposed by user interference. We examine this problem, stripped of obfuscating complexity in a synchronous, noise-free setting.

A misreading of a "folk theorem" to the effect that CDMA schemes degrade "gracefully" might suggest that the probability of receiver error increases smoothly (and slowly) as the number of users increases. Such a reading would be erroneous, however, and indeed, there is an asymptotically abrupt, catastrophic breakdown in performance at a critical rate of growth of the number of users with the number of chips in the pseudonoise signature sequence. This phenomenon is essentially combinatorial in nature and may be explicated by random graph methods. We begin by setting up notation.

II. Direct Sequence Spread Spectrum

Suppose there are K users in a direct sequence spread spectrum setting. The ith user is assigned a signature waveform

$$s_i(t) = \sum_{j=1}^{n} \sigma_{ij} p(t - jT_c) \qquad (0 \le t \le T)$$

where $\boldsymbol{\sigma}_i = (\sigma_{i1}, \dots, \sigma_{in})$ is a code sequence of n chips taking values in $\{-1, 1\}$, $p(t)$ is a pulse of energy $1/n$ with support in an interval $[-T_c, 0]$ where T_c is the chip interval, and $T = nT_c$ is the duration of the signature waveform, i.e., the bit transmission interval. Each signature waveform $s_i(t)$ is antipodally modulated by the bit $b_i \in \{-1, 1\}$ transmitted by the ith user. If the users transmit in synchrony, channel noise is abeyant, and the received amplitudes of the individual waveforms are the same then the received waveform is of the form

$$\begin{aligned} r(t) &= \sum_{i=1}^{K} A b_i s_i(t) \\ &= A \sum_{i=1}^{K} \sum_{j=1}^{n} b_i \sigma_{ij} p(t - jT_c) \qquad (0 \le t \le T). \end{aligned}$$

This antiseptic setting provides the clearest venue in which to examine user interference shorn of external complexity. As we will see shortly, there is no fundamental bar to the extension of the following results to more general settings, albeit at the cost of added notational complexity.

Now, it is easy to see that the likelihood function depends on the observations only through the outputs of a bank of matched filters (cf. Verdú [1])

$$Y_k \triangleq \int_0^T r(t) s_k(t)\, dt \qquad (1 \le k \le K).$$

It follows that $\mathbf{Y} = (Y_1, \dots, Y_K)$ is a sufficient statistic for demodulating the bit sequence $\mathbf{b} = (b_1, \dots, b_K)$.

Let us rewrite these equations in a more compact vector form. The received signal is the vector

$$\mathbf{r} = \frac{A}{\sqrt{n}} \sum_{i=1}^{K} b_i \boldsymbol{\sigma}_i$$

from which we form the sufficient statistic $\mathbf{Y}$ via the inner products

$$\begin{aligned} Y_k = \frac{1}{\sqrt{n}} \langle \mathbf{r}, \boldsymbol{\sigma}_k \rangle &= \frac{A}{n} \sum_{i=1}^{K} b_i \langle \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_k \rangle \\ &= \frac{A}{n} \sum_{i=1}^{K} \sum_{j=1}^{n} b_i \sigma_{ij} \sigma_{kj} \qquad (1 \le k \le K). \quad (*) \end{aligned}$$

In conventional single user detection the receiver forms the estimate $\widehat{b}_k$ of the information bit b_k according to

$$\widehat{b}_k = \operatorname{sgn} Y_k = \operatorname{sgn}\left(\sum_{i=1}^{K} \sum_{j=1}^{n} b_i \sigma_{ij} \sigma_{kj} \right) \qquad (1 \le k \le K)$$

and we shall restrict our attention to this setting while indicating briefly an extension to the decorrelating detector.

Suppose the signature sequences are randomly selected by independent sampling from the outcomes of symmetric Bernoulli trials, i.e., the sequence of random variables $\{\sigma_{ij}\}$ is i.i.d. with

$$\Pr\{\sigma_{ij} = -1\} = \Pr\{\sigma_{ij} = +1\} = 1/2.$$

Further suppose that the information bits $b_1, \dots, b_K$ can be arbitrary. How many users can be supported in such a scheme?

III. Fixed Fraction of Errors

We are interested in the number of receiver errors

$$\mathcal{E} = \sum_{i=1}^{K} 1_{\{\widehat{b}_i \neq b_i\}}.$$

Accordingly, for $k = 1, \dots, K$, define the random variables

$$\begin{aligned} X_k &\triangleq b_k \sum_{i=1}^{K} \sum_{j=1}^{n} b_i \sigma_{ij} \sigma_{kj} \\ &= n + \sum_{\substack{i=1 \\ i \neq k}}^{K} \sum_{j=1}^{n} b_k b_i \sigma_{ij} \sigma_{kj}. \qquad (**) \end{aligned}$$

Let $p = p(K, n) = \Pr\{\widehat{b}_i \neq b_i\}$ connote the probability of a single receiver error. Observe that the kth bit estimate is in error iff $X_k \leq 0$. Hence

$$p(K, n) = \mathbf{P}\{X_k \leq 0\} = \mathbf{P}\{S^{(k)}_{(K-1)n} \leq -n\} \qquad (***)$$

where

$$S^{(k)}_{(K-1)n} \triangleq \sum_{\substack{i=1 \\ i \neq k}}^{K} \sum_{j=1}^{n} b_k b_i \sigma_{ij} \sigma_{kj}$$

is a symmetric random walk over $(K-1)n$ steps. Now let the number of users $K = K_n$ depend explicitly on dimensionality n. If K_n increases at a suitable rate with n, we can now deploy sharp large deviation normal estimates for the tail of the binomial (cf. Feller's wonderful text [2]) to estimate the tail probability on the right-hand side of $(***)$.

Write $\Phi(\cdot)$ for the Gaussian distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

We hence obtain the following:

Theorem 1 *If $K = K_n$ grows with n such that $K_n/\sqrt{n} \to \infty$ as $n \to \infty$, then*

$$p(K_n, n) \sim \Phi\left(\frac{-\sqrt{n}}{\sqrt{K_n}}\right) \qquad (n \to \infty).$$

If, in addition, $K_n = o(n)$, then

$$p(K_n, n) \sim \frac{\sqrt{K_n}}{\sqrt{2\pi n}}\, e^{-n/2K_n} \qquad (n \to \infty).$$

But the expected number of receiver errors is given simply by

$$\mathbf{E}(\mathcal{E}) = \mathbf{E}\big(\mathcal{E}(K_n, n)\big) = K_n p(K_n, n).$$

We hence immediately have:

Corollary 1 *For any $0 < \gamma < 1/2$, if $K = K_n$ increases such that*

$$K_n \sim \frac{n}{\left[\Phi^{-1}(\gamma)\right]^2} \qquad (n \to \infty),$$

then $\mathbf{E}(\mathcal{E})/K_n \to \gamma$ as $n \to \infty$.

Thus, even if a constant fraction of errors can be tolerated, the number of users cannot increase faster than a constant times n. There is a catastrophic breakdown (a "phase transition") much earlier under the more stringent condition that no errors are permitted.

IV. No Errors: A Poisson Law

Under what conditions can we say more about the distribution of the number of receiver errors $\mathcal{E} = \mathcal{E}(K_n, n)$? It transpires that for a suitable rate of growth of K_n with n, the number of errors actually follows a Poisson law.

Theorem 2 *Fix any positive λ. If $K = K_n$ grows with n such that*

$$\begin{aligned} K_n = \frac{n}{2\log n} + \frac{n \log\log n}{4\log^2 n} + \frac{n \log(4\pi)}{4 \log^2 n} \\ + \frac{n \log\log\lambda}{2\log^2 n} + \mathcal{O}\left(\frac{n\log\log n}{\log^3 n}\right) \qquad (****) \end{aligned}$$

then the number of receiver errors $\mathcal{E} = \mathcal{E}(K_n, n)$ converges in distribution to $\mathrm{Po}(\lambda)$, *the Poisson distribution with parameter λ. In other words, for every nonnegative integer j,*

$$\Pr\{\mathcal{E} = j\} \to \frac{\lambda^j}{j!} e^{-\lambda} \qquad (n \to \infty).$$

Thus, asymptotically, the number of receiver errors is Poisson. In particular, for the rate of growth given in $(****)$, the probability $\Pr\{\mathcal{E} = 0\}$ that there are no receiver errors tends to $e^{-\lambda}$ as $n \to \infty$. We hence observe a catastrophic threshold function at $n/2\log n$.

Corollary 2 *Fix any $0 < \epsilon < 1$. Then:*

1. *If, for all sufficiently large n,*

$$K_n \leq (1-\epsilon)\frac{n}{2\log n},$$

then $\mathbf{P}\{\mathcal{E} = 0\} \to 1$ as $n \to \infty$.

2. *Conversely, if, for all sufficiently large n,*

$$K_n \geq (1+\epsilon)\frac{n}{2\log n},$$

then $\mathbf{P}\{\mathcal{E} = 0\} \to 0$ as $n \to \infty$.

Roughly speaking, *if K is less than $n/2\log n$, the matched filter decisions are guaranteed to be error free*; contrariwise, *if K exceeds $n/2\log n$ there are guaranteed to be errors*. The rather fine order of infinity in the asymptotics of $(****)$ actually allows us to deduce much more precise results.

Dependencies in the random variables X_k $(1 \leq k \leq K)$ defined in $(**)$ complicate the analysis. The method of proof of the theorem is to reduce the problem to the study of a random walk $\mathbf{S}_n$ where, for each n, $\mathbf{S}_n$ is the row sum of a triangular array of lattice random vectors. The principal technical result then needed to complete the Poissonisation argument is a sharp estimate for the probability of large deviations of the walk $\mathbf{S}_n$. The details are involved and the calculations delicate. For a large deviation theorem in this vein see Fang and Venkatesh [3].

V. Matched Filter Detector: Extensions

The basic approach may be extended, at some notational and computational cost, to include some of the other perturbants to the system that we had excluded. Embedding the waveforms in additive, white Gaussian noise does not create any new difficulties in analysis—the additive noise terms at the output of the matched filters are i.i.d., Gaussian, and we can run the analysis through by first conditioning on them and finally taking expectations to remove the conditioning. Different waveform amplitudes can likewise be handled if, for instance, the amplitudes are bounded between known limits or their distribution known. Again, one proceeds by conditioning.

VI. Decorrelating Detector

The decorrelating detector (cf. Verdú [1]) is an alternative approach to the matched filter detector. While its implementation complexity is higher, it has some very attractive properties among which is the fact that it does not require knowledge of the energies of any of the active users.

Let's first introduce some new notation. Form the $n \times K$ matrix

$$\boldsymbol{\Sigma} \triangleq \begin{bmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_2 & \dots & \boldsymbol{\sigma}_K \end{bmatrix}$$

whose components are all ± 1 and whose columns are identically the random signature sequences of each of the K users. We can now succinctly rewrite the sufficient statistic $\mathbf{Y}$ whose components are given by $(*)$ in matrix-vector form as[1]

$$\mathbf{Y} = \frac{A}{n} \left(\boldsymbol{\Sigma}^T \boldsymbol{\Sigma} \right) \mathbf{b}$$

where $\mathbf{b} = (b_1, b_2, \dots, b_K)^T$ is the (column) vector of data bits. The decorrelating detector forms the bit estimates $\widehat{\mathbf{b}} = \left(\widehat{b}_1, \widehat{b}_2, \dots, \widehat{b}_K\right)^T$ according to

$$\widehat{\mathbf{b}} = \operatorname{sgn}\left(\boldsymbol{\Sigma}^T \boldsymbol{\Sigma} \right)' \mathbf{Y},$$

where, $\left(\boldsymbol{\Sigma}^T \boldsymbol{\Sigma} \right)'$ denotes the Moore-Penrose generalised inverse of $\boldsymbol{\Sigma}^T \boldsymbol{\Sigma}$. Observe that

$$\left(\boldsymbol{\Sigma}^T \boldsymbol{\Sigma} \right)' = \left(\boldsymbol{\Sigma}^T \boldsymbol{\Sigma} \right)^{-1}$$

if the signature sequences are linearly independent. In this case, of course, we automatically recover all data bits without error. Komlós [4] has shown, however, that almost all square binary matrices are nonsingular asymptotically. Particularised to our purposes, we have the following result:

Lemma 1 *If $K = K_n \le n$, the probability that the random signature sequences $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots, \boldsymbol{\sigma}_K$ are linearly independent tends to 1 as $n \to \infty$.*

(Kahn, Komlós, and Szemeredi have improved Komlós' original result and have shown indeed that the probability that a random square binary matrix is singular is exponentially small.) This immediately leads us to:

Theorem 3 *If $K_n \le n$ then the decorrelating detector makes no errors with probability approaching one as $n \to \infty$ for any sequence of data bits. Conversely, if $K_n > n$, with probability bounded away from zero, there exist sequences of data bits which cannot all be recovered by the decorrelating detector.*

Somewhat more precise statements can also be made.

Noisy channels are harder to analyse for the decorrelating detector as the generalised inverse causes correlations in the noise variables. See Verdú [5] for details.

VII. Conclusion

User interference induces random graph-theoretic effects in direct sequence spread spectrum systems. These effects show up not gradually as one might perhaps have expected but abruptly in the form of threshold phenomena in the number of users that can be accomodated.

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References

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[1] Interpret all vectors as *column* vectors.