Contagion and Equilibria in Diversified Financial Networks

Victor Amelkin, Santosh S. Venkatesh, and Rakesh Vohra

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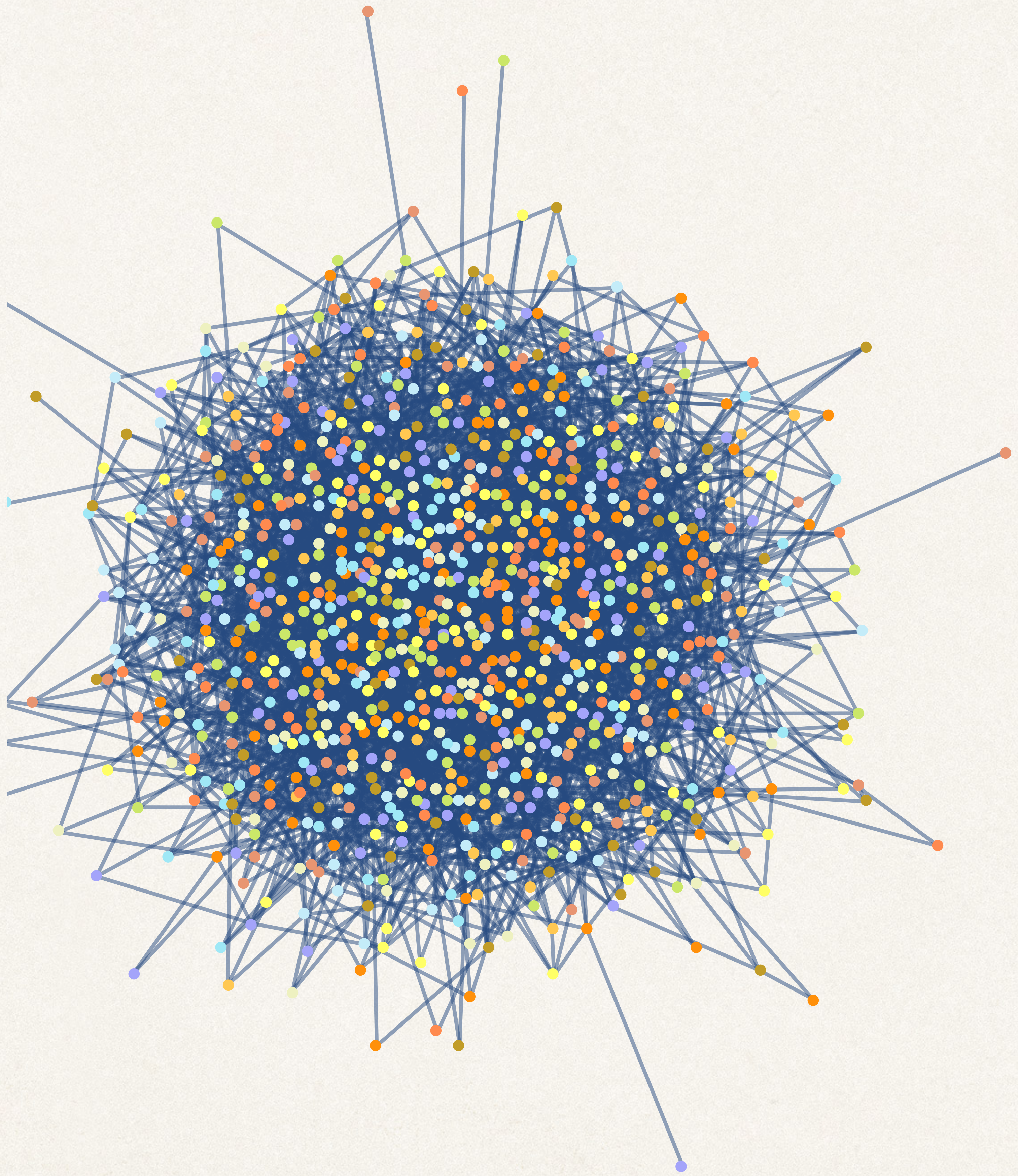
diversification

low high

conduits

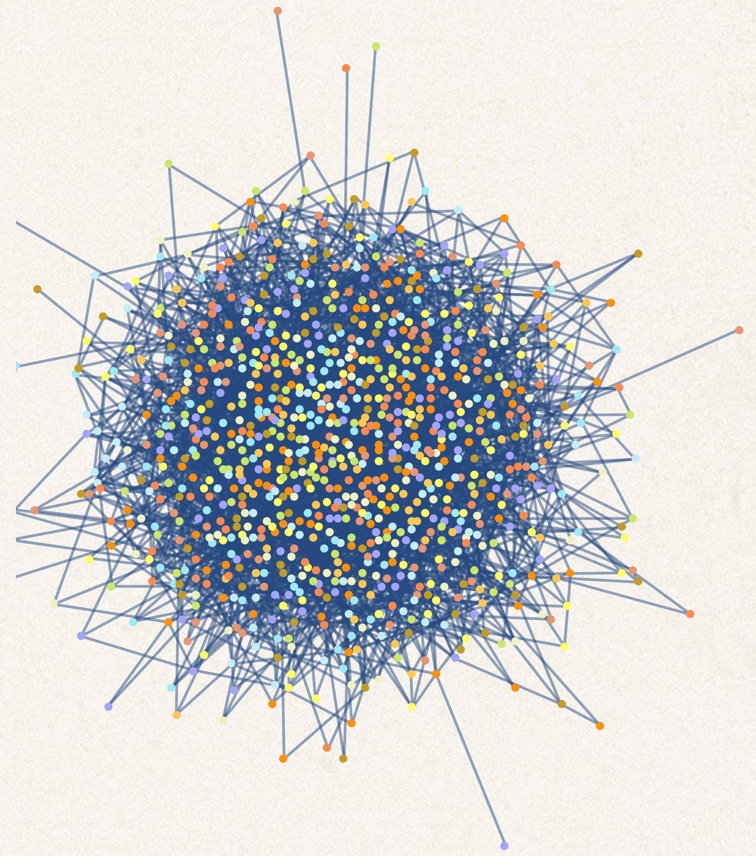
resilience



Challenges

Modelling diversification:

topological modelling: diversification captured by degree distribution



Absence of analytical closed form for equilibrium firm valuations:

focus on specific topologies

Firm valuation a non-linear function:

multiple equilibria

Model

Valuation

endowment

cross-shareholdings

default costs

Elliott, Golub, Jackson (2014)

n firms

$V_i :=$ valuation of firm i $\qquad$ $\mathbf{V} = (V_i)$

$e_i :=$ *endowment* of firm i $\qquad$ $\mathbf{e} = (e_i)$

$C_{ij} :=$ share of firm j held by firm i $\qquad$ $\mathbf{C} = [C_{ij}]$

$1 > C_{1j} + \cdots + C_{nj} =: c_j$ *exposure* of firm j

$\tau :=$ insolvency threshold $\qquad \mathbb{1}_{\{V_i \leq \tau\}} = \begin{cases} 1 & \text{if } V_i \leq \tau, \\ 0 & \text{if } V_i > \tau. \end{cases}$ $\qquad \mathbb{1}_{\{\mathbf{V} \leq \tau \mathbf{1}\}}$

$\beta :=$ default cost

Equilibria

$$V_i = e_i + \sum_{j=1}^{n} C_{ij} V_j - \beta \mathbb{1}_{\{V_i \leq \tau\}} \qquad \mathbf{V} = \mathbf{e} + \mathbf{C}\mathbf{V} - \beta \mathbb{1}_{\{\mathbf{V} \leq \tau \mathbf{1}\}}$$

The annoying sub-text: “book” versus “market” valuations

$$\mathbf{V} = \mathbf{V}_{\text{book}}$$

$$\mathbf{V}_{\text{market}} = \operatorname{diag}(1 - c_1, \ldots, 1 - c_n)\mathbf{V}_{\text{book}}$$

No interpretable analytical solutions except in special, very regular cases

Multiple equilibria

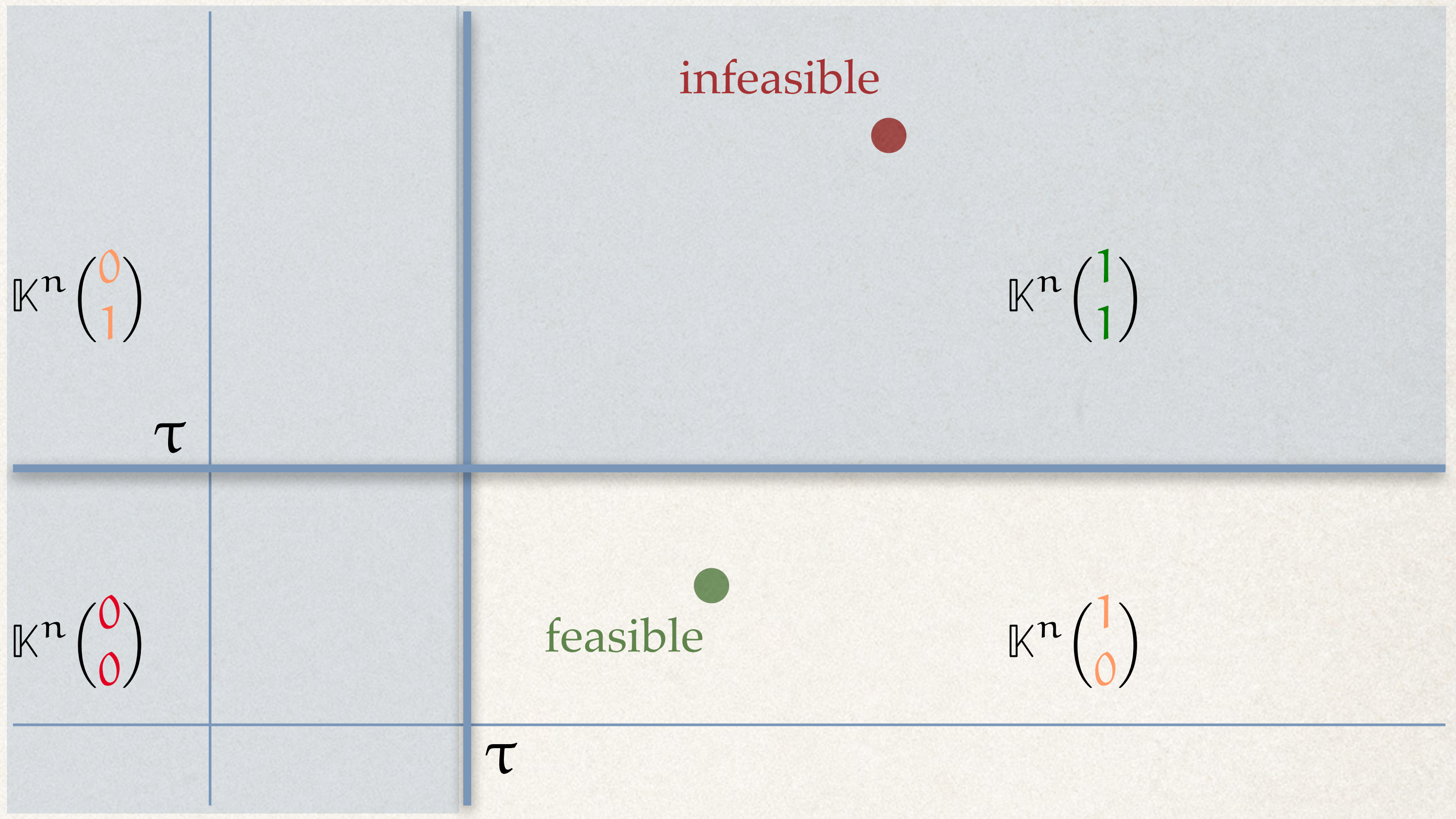
- compact lattice
- maximal and minimal equilibria

Putative and feasible equilibria

$$\mathbf{V} = \mathbf{e} + \mathbf{C}\mathbf{V} - \beta \mathbb{1}_{\{\mathbf{V} \leq \tau \mathbf{1}\}}$$

Putative solvency indicators

$$\mathbf{k} = (k_1, \ldots, k_n)^\intercal \in \{0, 1\}^n$$



Orthants

The orthant $\mathbb{K}^n(\mathbf{k})$ consists of points $\mathbf{x} = (x_1, \ldots, x_n)^\intercal$ in $\mathbb{R}^n$ satisfying $x_i > \tau$ if $k_i = 1$ and $x_i \leq \tau$ if $k_i = 0$.

Putative equilibria

$$\mathbf{V} = \mathbf{e} + \mathbf{C}\mathbf{V} - \beta(\mathbf{1} - \mathbf{k})$$

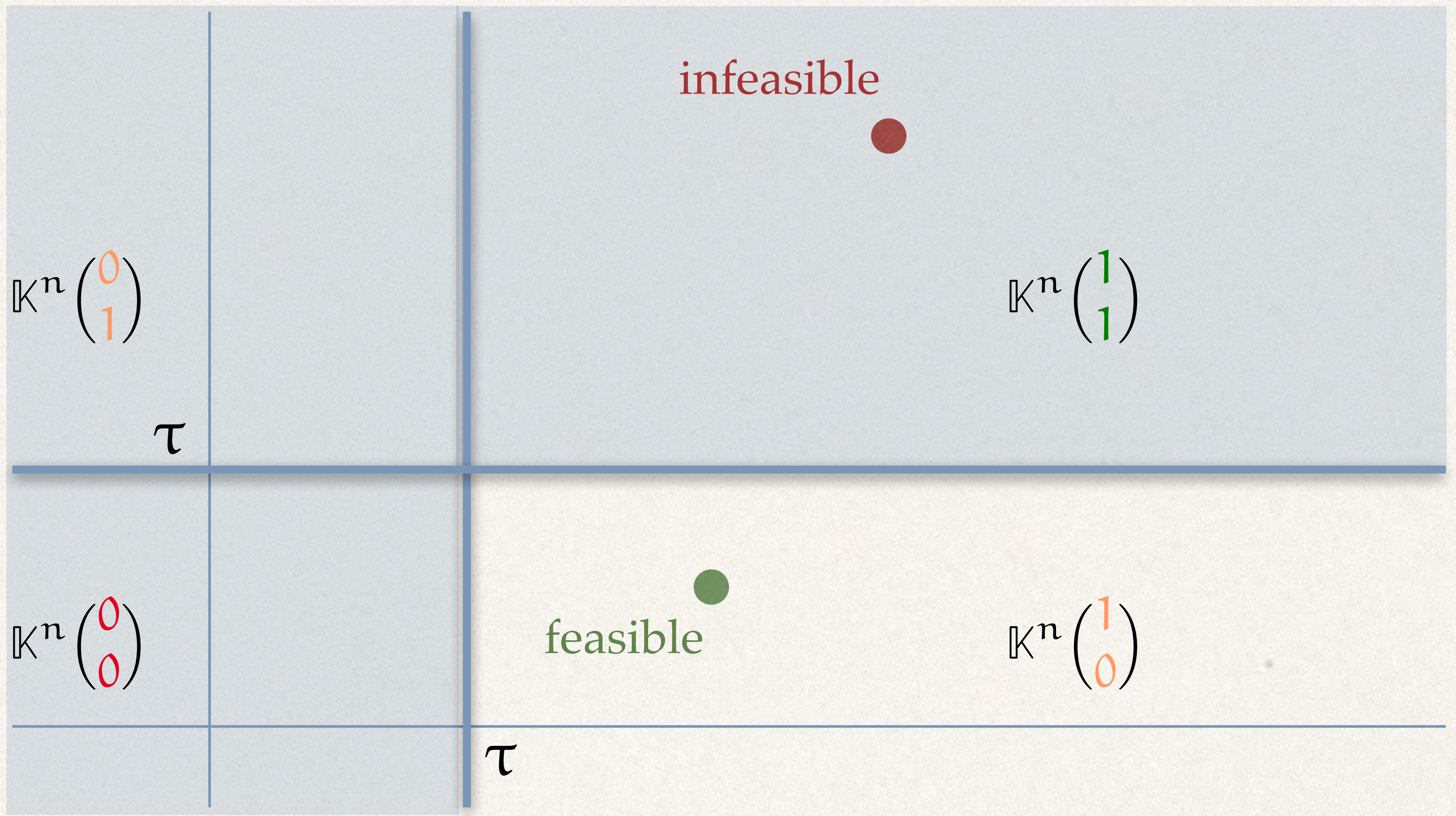
A putative equilibrium $\mathbf{V} = \mathbf{V}(\mathbf{k})$ is feasible
(for a putative solvency indicator $\mathbf{k}$) if, and only if, $\mathbf{V}(\mathbf{k}) \in \mathbb{K}^n(\mathbf{k})$.

Algebraic simplifications:

common endowment $\mathbf{e} = e\mathbf{1} = e(1, \ldots, 1)^\top$

common exposure $\mathbf{C} = c\mathbf{X} = c\begin{bmatrix}\mathbf{X}_1 & \cdots & \mathbf{X}_n\end{bmatrix}$

column stochastic matrix



non-linear fixed-point equation

$$\mathbf{V} = e\mathbf{1} + c\mathbf{X}\mathbf{V} - \beta\mathbb{1}_{\{\mathbf{V}\leq\tau\mathbf{1}\}}$$

putative linear fixed-point equations

$$\mathbf{V} = e\mathbf{1} + c\mathbf{X}\mathbf{V} - \beta(\mathbf{1} - \mathbf{k}) \qquad \left(\mathbf{k} \in \{0,1\}^n\right)$$

feasible if, and only if, $\mathbf{V} = \mathbf{V}(\mathbf{k}) \in \mathbb{K}^n(\mathbf{k})$

Our story in a slogan:
If, for any given exposure, the endowment reaches a critical level,
then *almost all* diversified networks are resilient to shocks and contagion.

Structure via randomisation

$$\mathbf{V} = e\mathbf{1} + c\mathbf{X}\mathbf{V} - \beta(\mathbf{1} - \mathbf{k})$$

column stochastic: available share distribution

Random cross-shareholding matrices

$$\mathbf{C} = c\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n\end{bmatrix} \qquad \mathbf{X}_j = \begin{pmatrix} X_{1j} \\ \vdots \\ X_{nj} \end{pmatrix} \qquad |\mathbf{X}_j| = X_{1j} + \cdots + X_{nj} = 1$$

Modelling diversification

Shares for each firm j are *exchangeable* random variables with column sum the common exposure c
Shares across firms are independent

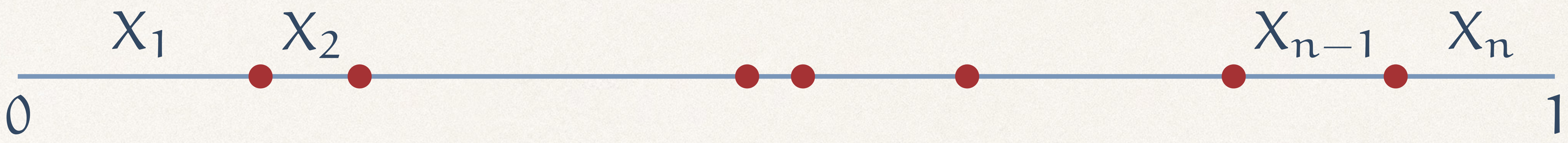
Encoding structure

Properties of distribution encode structure
Graph topology [degree, diameter, centrality] not immediately relevant

The de Finetti spacings

archetypal exchangeable system

$$\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$



$$\mathbf{P}\{X_1 > x_1, \ldots, X_n > x_n\} = \left[(1 - x_1 - \cdots - x_n)_+\right]^{n-1}$$

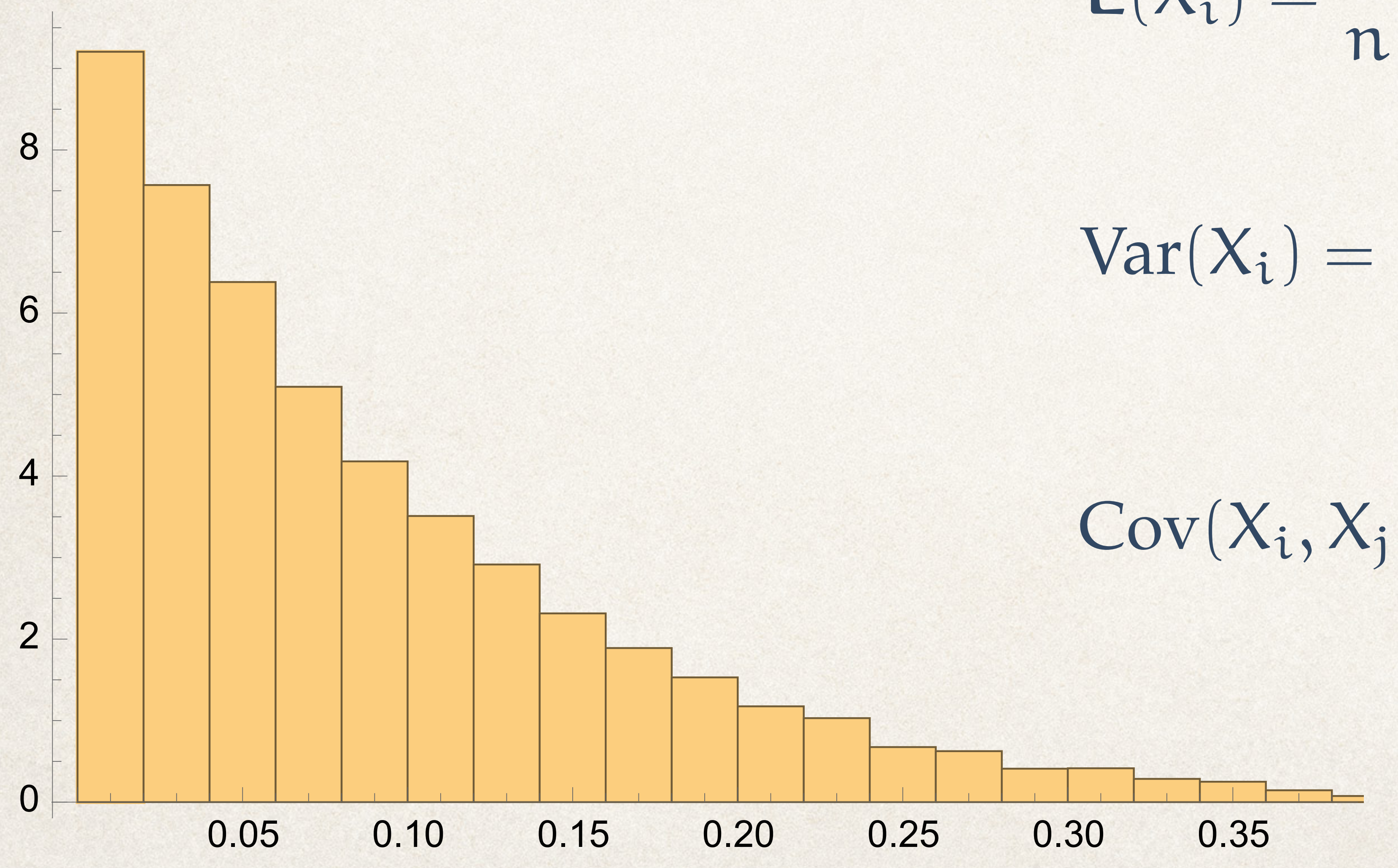
$$\mathrm{E}(X_i) = \frac{1}{n}$$

$$\mathrm{Var}(X_i) = \frac{n-1}{n^2(n+1)} \sim \frac{1}{n^2} \qquad \left\| X_i - \frac{1}{n} \right\|_p = \mathcal{O}\left(\frac{1}{n}\right)$$

} capturing full diversification

$$\mathrm{Cov}(X_i, X_j) = \frac{-1}{n^2(n+1)}$$

negatively correlated,
weak asymptotic dependence

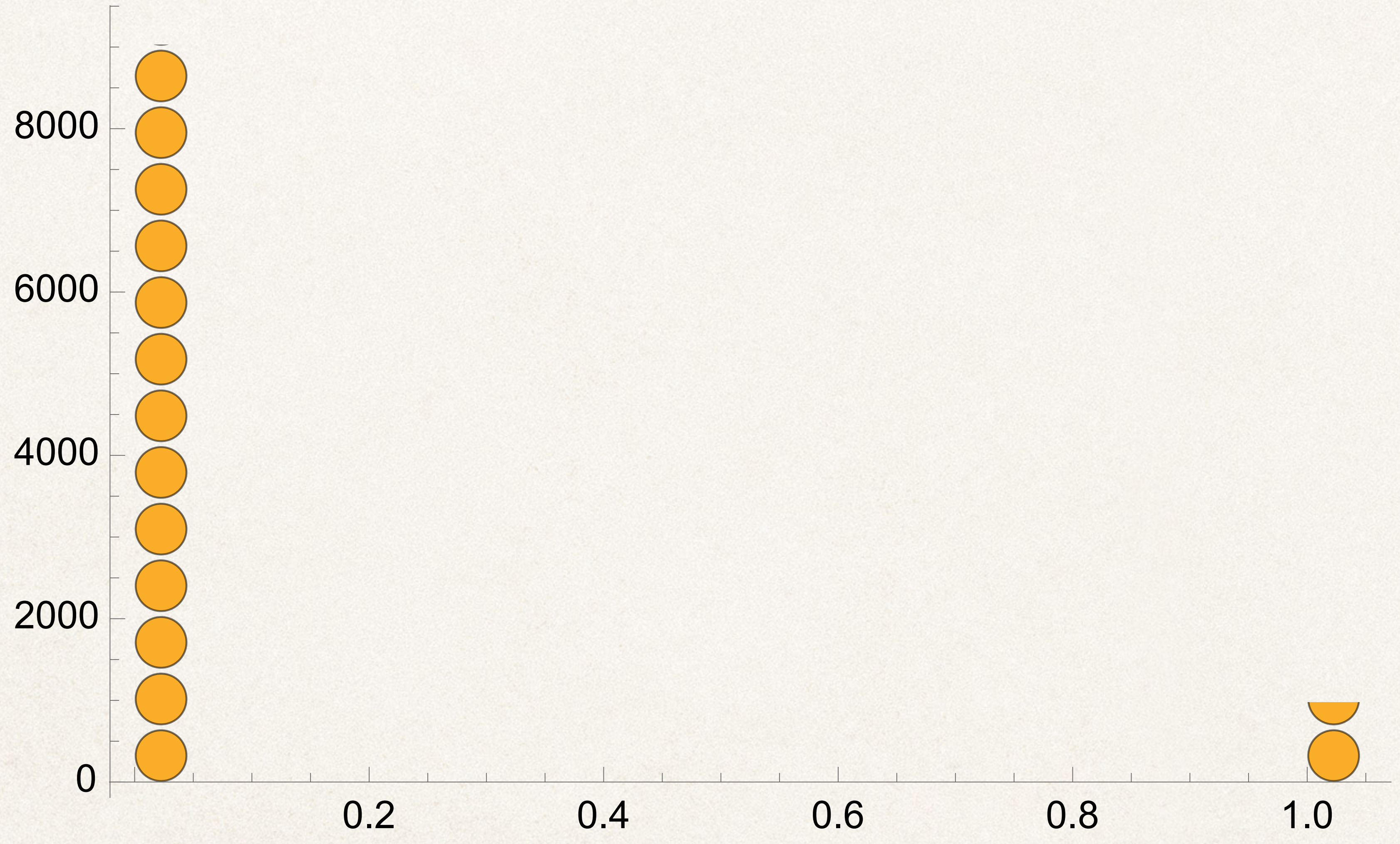


Coordinate spacings

pathological exchangeable system

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

The distribution $F_n(x_1, \ldots, x_n)$ of the spacings is atomic and places equal mass on each of the atoms $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, and $(0, 0, \ldots, 1)$.



$$X_i \sim \text{Bernoulli}(n^{-1})$$

$$E(X_i) = \frac{1}{n}$$

$$\text{Var}(X_i) = \frac{1}{n}\left(1 - \frac{1}{n}\right) \sim \frac{1}{n}$$

$$\left\| X_i - \frac{1}{n} \right\|_p = \mathcal{O}\left(\frac{1}{n^{\frac{1}{p}}}\right)$$

dependency structure

$$\text{Cov}(X_i, X_j) = -\frac{1}{n^2}$$

Asymptotically diffuse distributions

the essence of diversification

$$\mathbf{X}^{(n)} = \mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

spacings of the unit interval

$$X_i \geq 0 \qquad X_1 + \cdots + X_n = 1$$

exchangeable components

$$F_n(\mathbf{x}) = F_n(\Pi \mathbf{x}) \qquad \text{(all permutations } \Pi \mathbf{x} = (\Pi x_1, \ldots, \Pi x_n))$$

$$\mathsf{E}(X_i) = \frac{1}{n}$$

asymptotically diffuse condition

$$\left\| X_i - \frac{1}{n} \right\|_8 = \mathcal{O}\left(\frac{1}{n}\right)$$

Equilibria for a random matrix

$$\mathbf{V} = e\mathbf{1} + \mathbf{C}\mathbf{V} - \beta(\mathbf{1} - \mathbf{k})$$

Random matrix

$$\mathbf{C} = \mathbf{C}^{(n)} = c\begin{bmatrix} \mathbf{X}_1^{(n)} & \mathbf{X}_2^{(n)} & \cdots & \mathbf{X}_n^{(n)} \end{bmatrix}$$

column $\mathbf{X}_j^{(n)} = (X_{1j}^{(n)}, \ldots, X_{nj}^{(n)})^\intercal$

components: non-negative valued, exchangeable, asymptotically diffuse

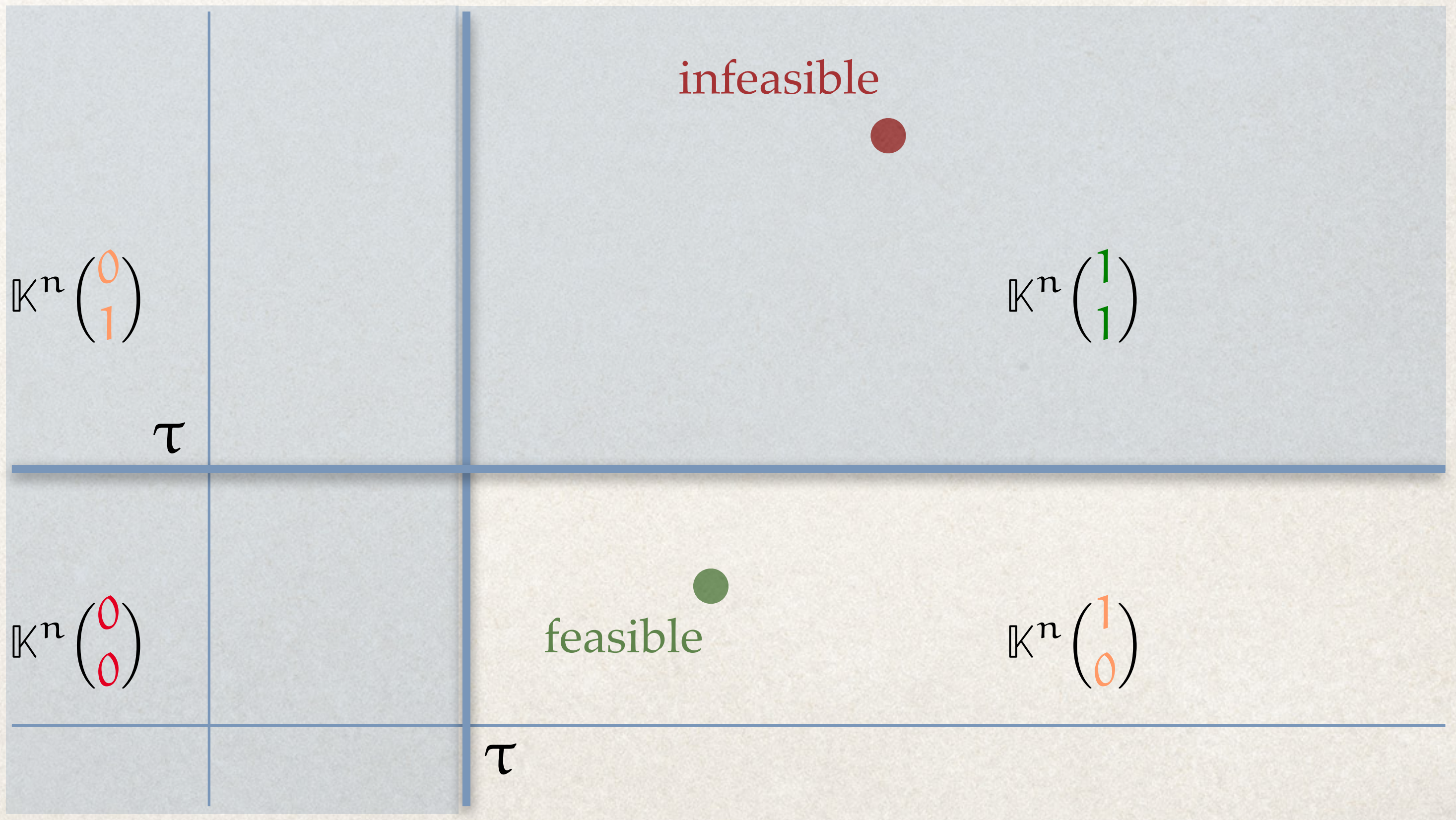
$$E(X_{ij}^{(n)}) = \frac{1}{n}$$

columns $\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)}$: independent, identically distributed

Putative equilibria

$$\mathbf{V}^{(n)}(\mathbf{k}^{(n)}) = \mathbf{V}(\mathbf{k}) = (\mathbf{I} - \mathbf{C})^{-1}\big((e-\beta)\mathbf{1} + \beta\mathbf{k}\big) \qquad \big(\mathbf{k}^{(n)} = \mathbf{k} \in \{0,1\}^n\big)$$

feasible if, and only if, $\mathbf{V}(\mathbf{k}) \in \mathbb{K}^n(\mathbf{k})$



Candidate equilibria

$$\mathbf{V} = e\mathbf{1} + \mathbf{C}\mathbf{V} - \beta(\mathbf{1} - \mathbf{k})$$

The regular clique

$$\overline{\mathbf{C}}^{(n)} = \overline{\mathbf{C}} := \tfrac{c}{n}\mathbf{1}\mathbf{1}^{\mathsf{T}} = \begin{bmatrix} \frac{c}{n} & \cdots & \frac{c}{n} \\ \cdots & \cdots & \cdots \\ \frac{c}{n} & \cdots & \frac{c}{n} \end{bmatrix}$$

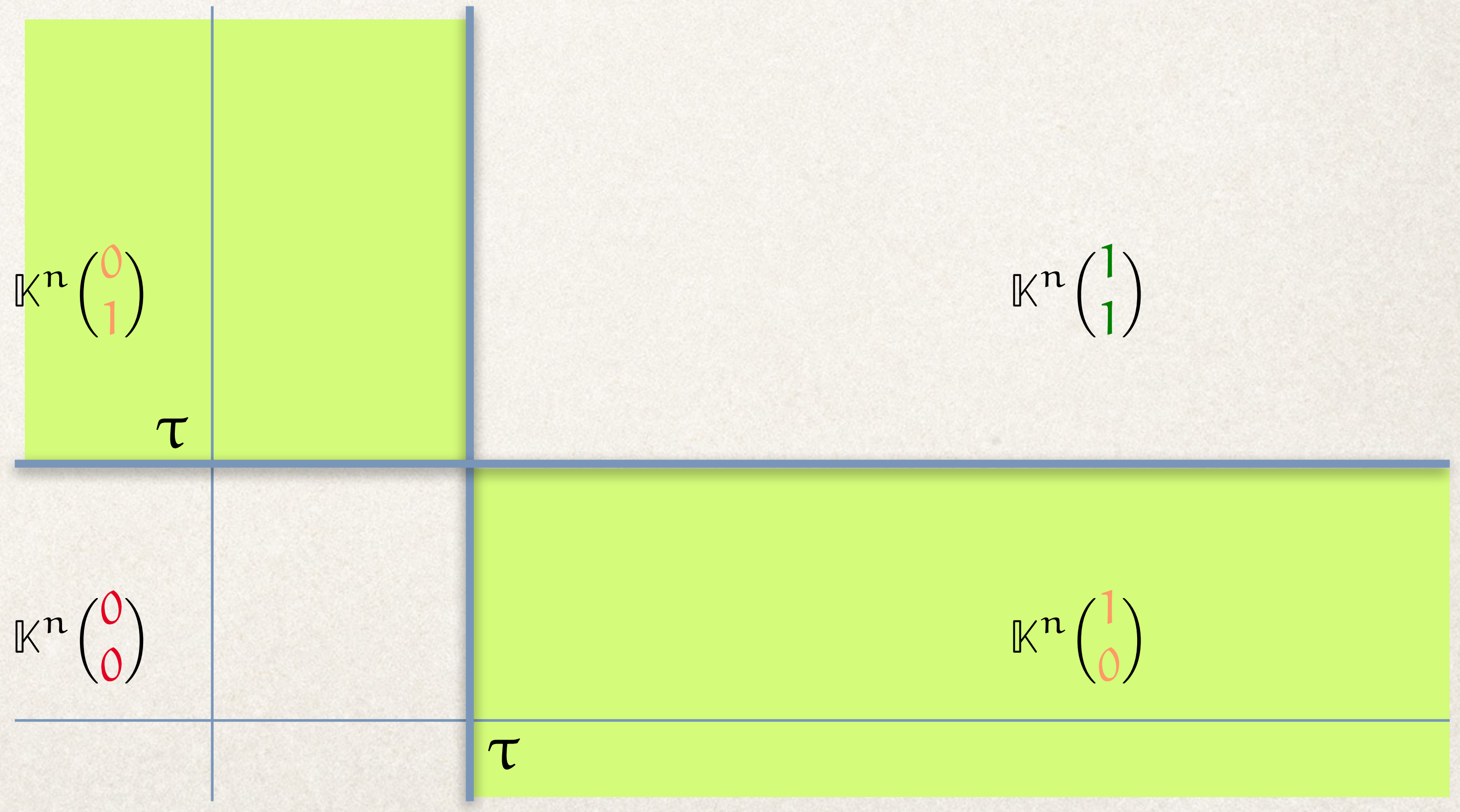
Putative equilibria

$$\overline{\mathbf{V}}^{(n)}(\mathbf{k}^{(n)}) = \overline{\mathbf{V}}(\mathbf{k}) = (\mathbf{I} - \overline{\mathbf{C}})^{-1}\left((e-\beta)\mathbf{1} + \beta\mathbf{k}\right) \qquad \left(\mathbf{k}^{(n)} = \mathbf{k} \in \{0,1\}^n\right)$$

feasible if, and only if, $\overline{\mathbf{V}}(\mathbf{k}) \in \mathbb{K}^n(\mathbf{k})$

explicit solutions

equivalence classes of solvency orthants determined upto permutations by $|\mathbf{k}|$



Concentration

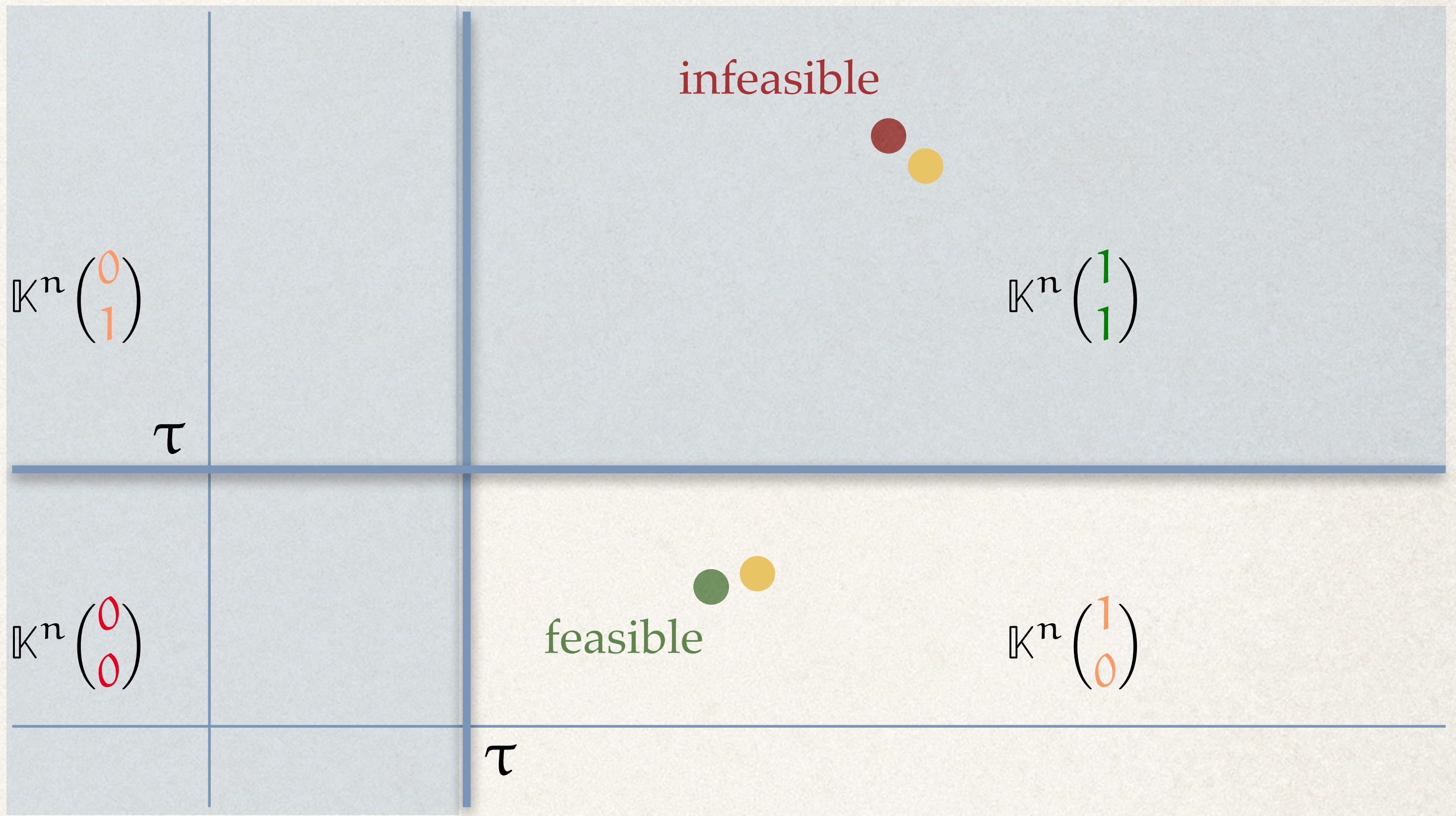
$$\mathbf{V}^{(n)}\left(\mathbf{k}^{(n)}\right) = \mathbf{V}(\mathbf{k}) = (\mathbf{I} - \mathbf{C})^{-1}\left((e-\beta)\mathbf{1} + \beta\mathbf{k}\right)$$

random cross-shareholding matrix

$$\mathbf{C} = \mathbf{C}^{(n)} = c\left[\mathbf{X}_1^{(n)} \quad \mathbf{X}_2^{(n)} \quad \cdots \quad \mathbf{X}_n^{(n)}\right]$$

regular clique

$$\overline{\mathbf{C}}^{(n)} = \overline{\mathbf{C}} := \frac{c}{n}\mathbf{1}\mathbf{1}^\mathsf{T} = \begin{bmatrix} \frac{c}{n} & \cdots & \frac{c}{n} \\ \cdots & \cdots & \cdots \\ \frac{c}{n} & \cdots & \frac{c}{n} \end{bmatrix}$$



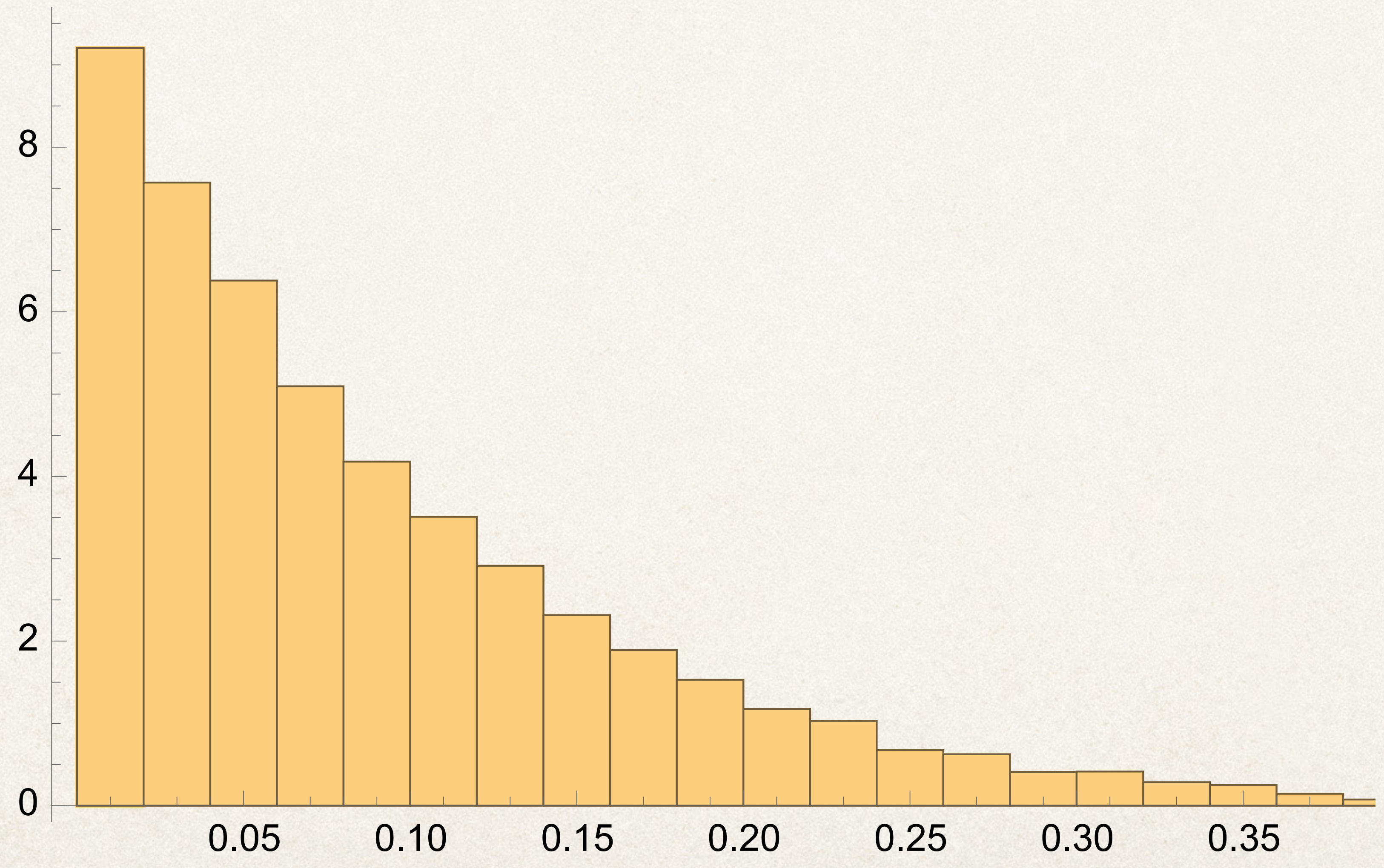
Theorem For any sequence of index vectors $\{\, \mathbf{k}^{(n)} \in \{0,1\}^n, n \geq 1 \,\}$, we have

$$\sup_{1\leq i\leq n} \left| V_i^{(n)}\left(\mathbf{k}^{(n)}\right) - \overline{V}_i^{(n)}\left(\mathbf{k}^{(n)}\right)\right| \to 0$$

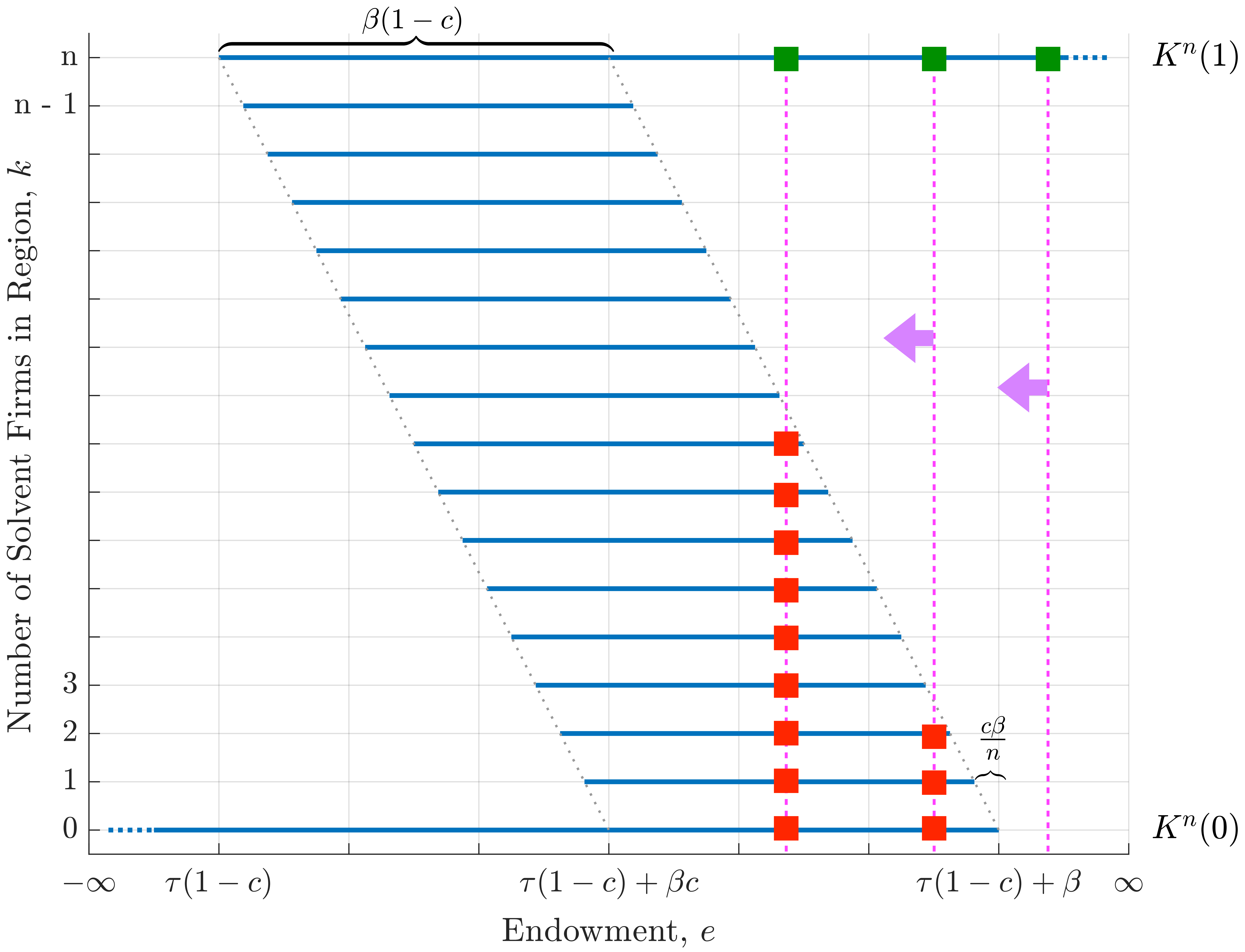
almost surely as $n \to \infty$.

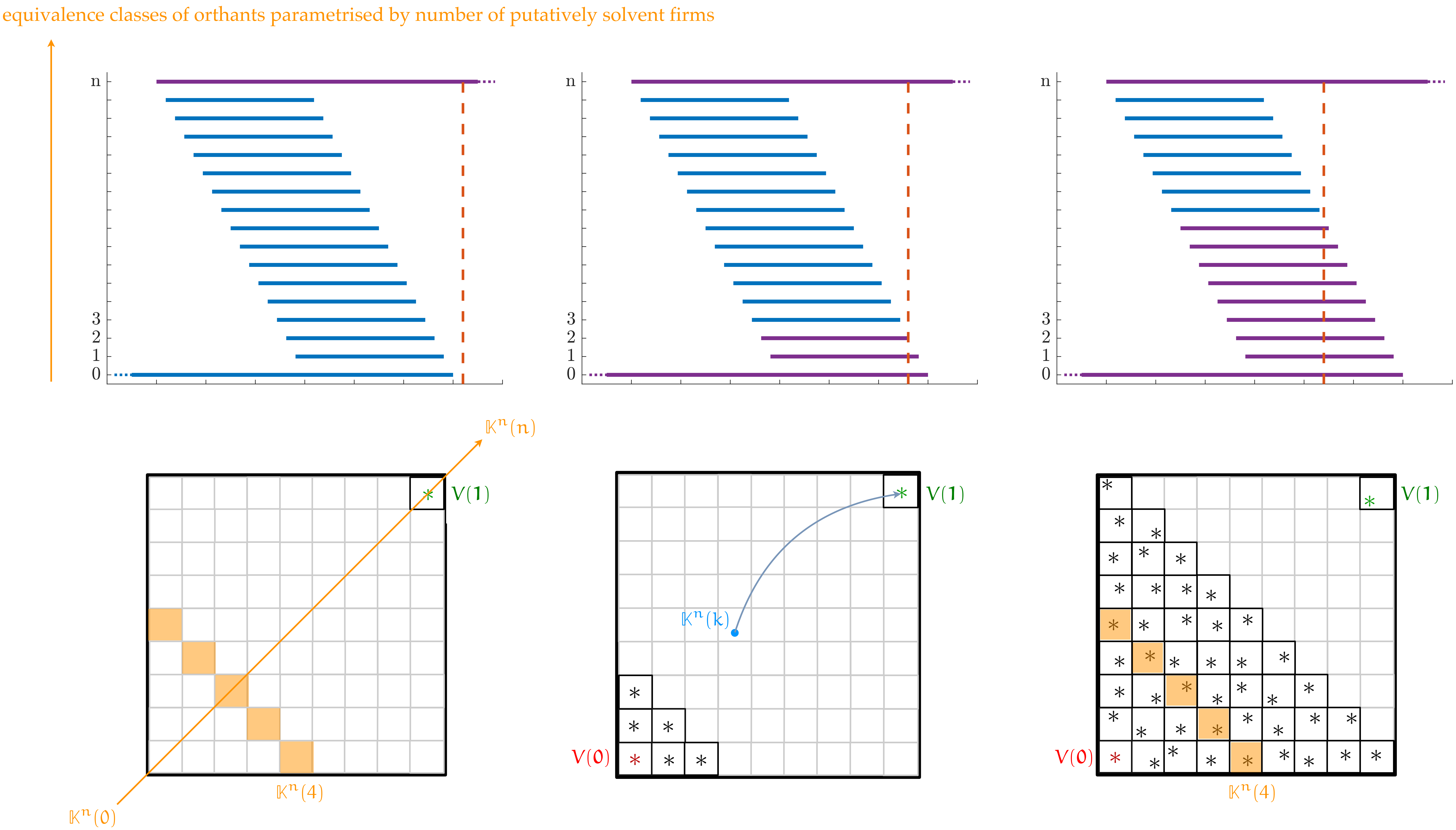
Slogan

The putative equilibria (*a fortiori* the feasible equilibria) of the random cross-shareholding matrix $\mathbf{C}$ are everywhere close to the corresponding equilibria of the regular clique $\overline{\mathbf{C}}$.



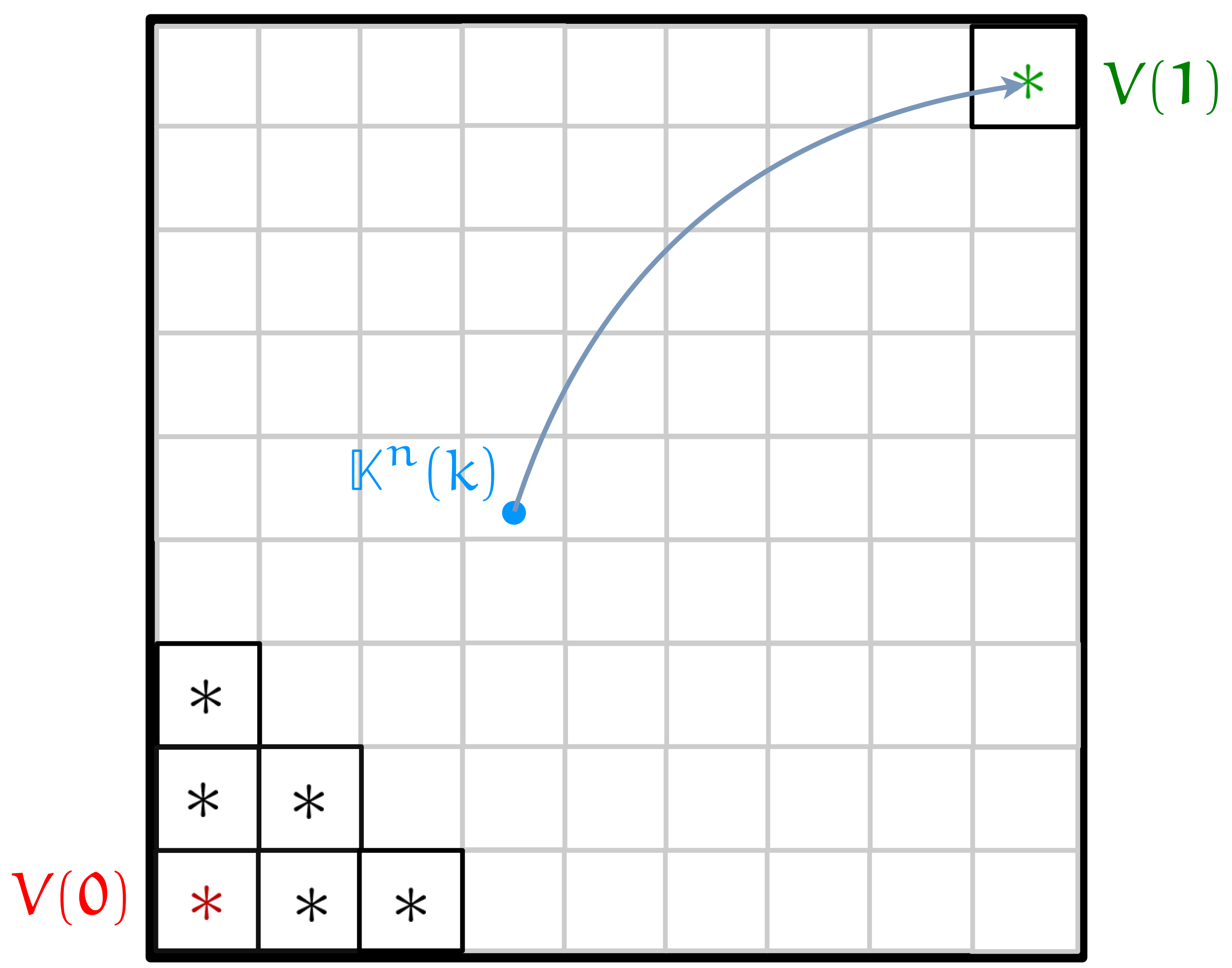
Feasibility regions for the regular clique





equivalence classes of orthants parametrised by number of putatively solvent firms
n
3
2
1
0
$\mathbb{K}^n(n)$
V(1)
$\mathbb{K}^n(4)$
$\mathbb{K}^n(0)$
$\mathbb{K}^n(k)$
V(0)

Response to shocks



Fictitious dynamic $\quad \mathbf{V}_{t+1} = \mathbf{e} + \mathbf{C}\mathbf{V}_t - \beta \mathbb{1}_{\{\mathbf{V}_t \leq \tau \mathbf{1}\}}$

Valuation shock

starting from the best (maximal) equilibrium suppose $n - k$ firms become insolvent

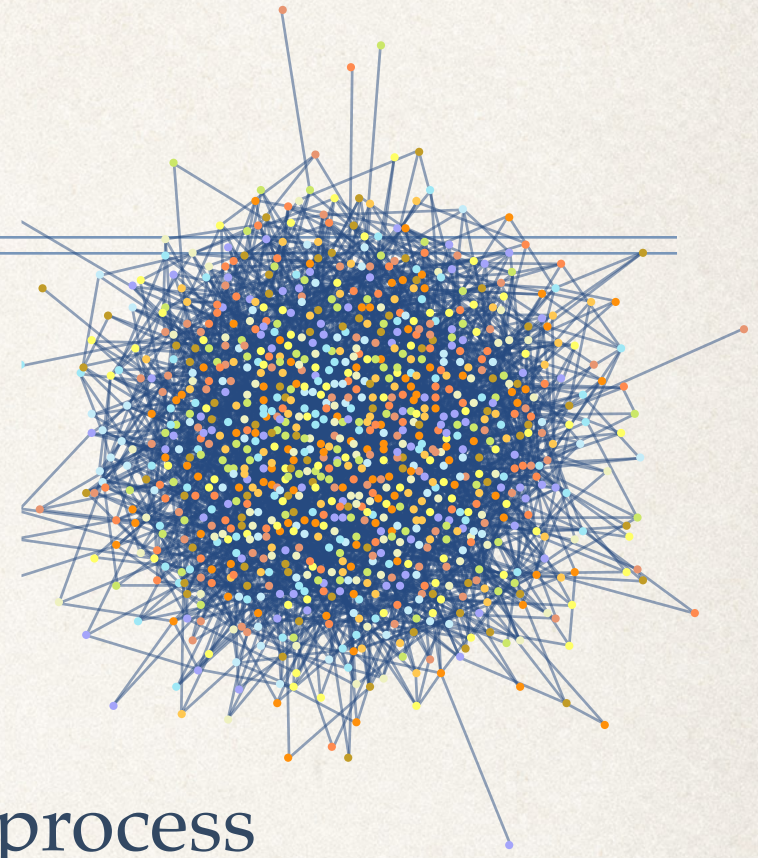
Fixed exposure: if the endowment is at or above a *critical value* then full recovery is assured

Fixed endowment: stability improves as exposure increases

Our story in a slogan:
If, for any given exposure, the endowment reaches a critical level,
then *almost all* diversified networks are resilient to shocks and contagion.

Quo vadis?

Extensions



Folding in topological graph structure

Erdös–Rényi digraphs $G_{n,p}$: out-degree of vertex j determines firms who hold shares in firm j's equity

Random matrix allocation: given exposure c, allocate j's shares via an asymptotically diffuse exchangeable process

Topological regular clique: assign shares equally to all j's neighbours

Multi-type random graphs, stochastic block models

Core-periphery networks, cross-border relations

Almost all instances of the topological random share matrix behave like the topological regular clique

No sensitivity to diversification, even for very small p

But we have no results in the *very* sparse domain when $d = np = \mathcal{O}(1)$ is small

Graphons, optimal bailouts [with Krishna Dasaratha and Rakesh Vohra]