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Random Interactions in Higher Order Neural Networks

Pierre Baldi and Santosh S. Venkatesh

***Abstract*—Recurrent networks of polynomial threshold elements with random symmetric interactions are studied. Precise asymptotic estimates are derived for the expected number of fixed points as a function of the margin of stability. In particular, it is shown that there is a critical range of margins of stability (depending on the degree of polynomial interaction) such that the expected number of fixed points with margins below the critical range grows exponentially with the number of nodes in the network, while the expected number of fixed points with margins above the critical range decreases exponentially with the number of nodes in the network. The random energy model is also briefly examined and links with higher order neural networks and higher order spin glass models made explicit.**

***Index Terms*— Neural networks, spin glasses, polynomial threshold elements, fixed points, Laplace's method.**

I. INTRODUCTION

Recurrent networks of formal neurons have been popular in a variety of computational applications. The model neurons in such structures are typically linear threshold elements which compute the sign of a linear form of the inputs. A recurrent network results when such elements are fully interconnected, and as in any recurrent system, the fixed points are important in the characterization of the computations done by the structure. A particular case of interest results when the interconnections between neurons are symmetric: in such cases network dynamics are regulated by a Hamiltonian or energy function (cf. Hopfield [1], for instance). In such an instance, we can imagine the state space of the network to be embedded in an energy landscape with fixed points residing at energy minima. A classical application of such networks is in associative memory where neural interactions are adjusted so that memories are stored as local attractors.

We consider here a natural extension of the model to recurrent networks comprised of higher order neurons that compute the sign of a *polynomial* form of the inputs. The added degrees of freedom in specifying the polynomial interaction coefficients can be expected to enrich the computational dynamics that result. Distinct features emerge, however, in the analysis of these structures depending on whether the higher order interactions are programmed (or "learned") or random.

In the programmed scenario, the goal is to tailor the higher order interaction coefficients so as to obtain desired dynamical behaviors; this leads naturally to questions of capacity and efficiency of higher order networks of given degree of polynomial interaction. In two concurrent papers [2], [3], we present rigorous results on algorithmic capacity and efficiency in programmed situations for higher order networks (cf. also Newman [4]). The main results can be summarized briefly as follows: the computational gains in higher order networks parallel the extra degrees of freedom in specifying the polynomial interaction coefficients; in particular, regardless of the algorithm used to specify the interaction coefficients, the information storage capability of a higher order network is of the order of one bit per interaction coefficient.

Higher order systems where the polynomial interactions are random may be useful as models of disordered systems in statistical physics (spin glasses), or of neural networks, before any learning has occurred, or in the limiting case when too much learning has occurred (the onset of senility!). These will be our focus of analysis in this paper: in particular, we consider recurrent, higher order neural networks with symmetric, random polynomial interactions. We characterize the fixed points of these structures according to their *margin of stability*[1] that is a measure of how stable a fixed point is with respect to perturbations. Our main result may be informally stated as follows:

> There exists a critical range of margins of stability (depending on the degree of polynomial interaction) such that the expected number of fixed points with margins of stability below the critical range increases exponentially in the size of the network while the expected number of fixed points with margins of stability above the critical range decays exponentially as the size of the network is increased.

There is thus a threshold phenomenon in evidence for the expected number of fixed points around the critical range of the margin of stability. The fact that for a certain range of margins the expected number of fixed points grows exponentially with the number of nodes in the network is not unexpected; more counter-intuitive, perhaps, is the existence of a critical margin of stability above which the expected number of fixed points actually decays as more nodes are added. We also provide exact asymptotic expressions for the coefficients and exponents in the regime of exponential behavior, and evaluate the critical margins of stability. While considerable attention has been focused on spin glass models in the statistical physics literature, at

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[1] In this context, this notion is due to Komlós and Paturi [9].

the time of writing rigorous results appear to have been confined to the case of linear interactions and to estimates of the expectation of the *total* number of fixed points (cf. Edwards and Tanaka [5], Gross and Mezard [6], and McEliece and Posner [7]). The estimates derived here provide a finer partition of the expected number of fixed points grouped according to their margins of stability, and extend the results to higher order cases with polynomial interactions where the statistical dependences grow more acute.

The basic analytical tool used is Laplace's method for integrals. The assumed random, independent, and symmetric nature of the interactions makes for some simplicity in analysis. The results derived here for the disordered case may also give some intuition in programmed situations where the interaction dependences are weak, though a corresponding analysis of the number of fixed points for the programmed case is typically more complicated. The analysis for the programmed case depends strongly on the algorithm of choice, and is made harder by the presence of statistical dependences in the interaction coefficients, especially in higher order cases. Rigorous estimates for the total number of fixed points are known only for the case of linear interactions programmed with the outer product algorithm: McElience *et al.* [8] conjectured based on the corresponding situation with random interactions that the number of extraneous fixed points is exponential in the number of nodes and this was rigorously shown by Komlós and Paturi [9]. The issue remains open for other algorithms such as the spectral algorithm (cf. Venkatesh and Psaltis [10]) even for the linear interaction case. For the higher order cases no formal results have been shown for any algorithm.

In Section II, we formally introduce recurrent higher order networks, and make precise the notion of the margin of stability of a fixed point. We also show that when the polynomial interactions are symmetric, network dynamics are regulated by an energy function. In Section III, we consider random, homogeneous, higher order networks and prove our main results. In Section IV, we deal with nonhomogeneous networks, and also briefly examine a different model of randomness known in the literature as the random energy model (cf. Derrida [11]). The proofs of the main theorems are developed in the body of the correspondence, while technical lemmas and calculations are confined to the two appendixes.

A brief word on notation: in addition to standard asymptotic conventions, we write $x_n \underset{\sim}{<} y_n$ if $x_n \leq y_n$ for n large enough; all logarithms in the exposition are to base e; we also denote by $\mathbf{B}$ the set $\{-1, 1\}$.

II. Polynomial Threshold Networks

We consider systems of n densely interacting threshold units each of which yields an instantaneous state -1 or $+1$. (This corresponds in the literature to a system of n Ising spins, or alternatively, a system of n neural states.) The state space is, hence, the set of vertices of the hypercube. We will in this discussion also restrict our attention throughout to *symmetric interaction systems.*

Let $\mathcal{I}_d$ be the family of all subsets of cardinality $d+1$ of the set $\{1, 2, \cdots, n\}$. Clearly $|\mathcal{I}_d| = \binom{n}{d+1}$. For any subset I of $\{1, 2, \cdots, n\}$, and for every state $\boldsymbol{u} = (u_1, u_2, \cdots, u_n) \in \mathbf{B}^n$, set $u_I = \prod_{i \in I} u_i$.

Definition 1: A homogenous polynomial threshold network of degree d is a network of n threshold elements with interactions specified by a set of $\binom{n}{d+1}$ real coefficients w_I indexed by I in $\mathcal{I}_d$. The evolution rule is asynchronous, and for $i \in \{1, \cdots, n\}$ is given by

$$u_i^+ = \operatorname{sgn}\left(\sum_{I \in \mathcal{I}_d : i \in I} w_I u_{I \setminus \{i\}}\right). \tag{1}$$

A *nonhomogeneous polynomial threshold network* of degree d is a network of n threshold elements with interactions specified by a set of $\sum_{j=1}^{d} \binom{n}{j+1}$ real coefficients w_I indexed by I in $\bigcup_{j=1}^{d} \mathcal{I}_j$, and, for $i \in \{1, \cdots, n\}$, the asynchronous evolution rule

$$u_i^+ = \operatorname{sgn}\left(\sum_{j=1}^{d} \sum_{I \in \mathcal{I}_j : i \in I} w_I u_{I \setminus \{i\}}\right). \tag{2}$$

These networks are readily seen to be natural generalizations to higher order of the familiar case of *linear threshold networks* ($d = 1$). These systems can be identified either with higher order spin glasses at zero temperature, or higher order neural networks. Starting from an arbitrary configuration or state, the system evolves asynchronously by a sequence of single "spin" flips involving spins which are misaligned with the instantaneous "field." The dynamics of these symmetric higher order systems are regulated by higher order extensions of the classical quadratic Hamiltonian. We define the *homogeneous Hamiltonian of degree* d by

$$H_d(\boldsymbol{u}) = -\sum_{I \in \mathcal{I}_d} w_I u_I. \tag{3}$$

In like fashion, we define the *nonhomogeneous Hamiltonian of degree* d by

$$\hat{H}_d(\boldsymbol{u}) = -\sum_{j=1}^{d} \sum_{I \in \mathcal{I}_d} w_I u_I. \tag{4}$$

We briefly sketch the proof of the following results (see also Maxwell *et al.* [12].)

Proposition 1: The functions H_d and $\hat{H}_d$ are nonincreasing under the evolution rules (1) and (2), respectively.

Proof: We consider the case of H_d, the nonhomogeneous case being similar. Assume $\boldsymbol{u} \mapsto \boldsymbol{u}^+$ is a mapping along some arbitrary trajectory in state space for a homogeneous polynomial threshold network of degree d. The proposition is trivially true if $\boldsymbol{u}$ is a fixed point. Consider the case where $\boldsymbol{u}$ and $\boldsymbol{u}^+$ differ only in a single component. Without loss of generality, assume the ith component of $\boldsymbol{u}$ changes sign: $u_i^+ = -u_i$ and $u_j^+ = u_j$ if $j \neq i$. Now consider $\delta H_d(\boldsymbol{u}) = H_d(\boldsymbol{u}^+) - H_d(\boldsymbol{u})$. Factoring out u_i in (3) we can write

$$H_d(\boldsymbol{u}) = -u_i \sum_{I \in \mathcal{I}_d : i \in I} w_I u_{I \setminus \{i\}} - \sum_{I \in \mathcal{I}_d : i \notin I} w_I u_I.$$

Hence,

$$\delta H_d(\boldsymbol{u}) = 2 u_i \sum_{I \in \mathcal{I}_d : i \in I} w_I u_{I \setminus \{i\}}.$$

By assumption, we have $u_i = -u_i^+ = -\operatorname{sgn}(\sum_{I \in \mathcal{I}_d : i \in I} w_I u_{I\{i\}})$, so that $\delta H_d(\boldsymbol{u}) \leq 0$. □

In the terminology of spin glasses, the state trajectories of these higher order networks can be seen to be following essentially a zero-temperature Monte Carlo (or Glauber) dynamics. Because of the monotonicity of the Hamiltonians given by (3) and (4) under the asynchronous evolution rule (1) [resp., (2)] the system always reaches a stable state (fixed point) where the relation (1) [resp., (2)] is satisfied with $u_i^+ = u_i$ for each of the n spins or neural states.

Definition 2: Let $\mathcal{B}$ be a nonnegative parameter (possibly depending on n). A fixed point $\boldsymbol{u} \in \mathbf{B}^n$ of a homogeneous polynomial threshold network of degree d is $\mathcal{B}$*-stable* if it satisfies

$$u_i \sum_{I \in \mathcal{I}_d : i \in I} w_I u_{I \setminus \{i\}} > \mathcal{B}, \qquad i = 1, \cdots, n. \tag{5}$$

In like fashion, a $\mathcal{B}$-stable state, $\boldsymbol{u} \in \mathbf{B}^n$, of a nonhomogeneous polynomial threshold network of degree d satisfies

$$u_i \sum_{j=1}^{d} \sum_{I \in \mathcal{I}_j : i \in I} w_I u_{I \setminus \{i\}} > \mathcal{B}, \qquad i = 1, \cdots, n. \tag{6}$$

For $\mathcal{B}$-stable states, $\mathcal{B}$ represents the *margin of stability* for the fixed point; we henceforth refer to $\mathcal{B}$ as the *margin*. We would expect $\mathcal{B}$-stable states with large margins to tend to exhibit correspondingly large basins of attraction, i.e., to be stable with respect to relatively large perturbations. Note that according to this definition, all fixed points are 0-stable states. Komlós and Paturi [9] utilize a similar notion in an analysis of the extraneous stable states for the case of a linear interaction network ($d = 1$) programmed using the outer product algorithm.

III. Homogeneous Networks

A. Higher Order Spin Glasses

Consider homogeneous polynomial threshold networks whose weights w_I, $I \in \mathcal{I}_d$ are i.i.d., $\mathcal{N}(0, 1)$ random variables. This is a natural generalization to higher order of Ising spin glasses with Gaussian interactions. We will derive an asymptotic estimate for the expected number of $\mathcal{B}$-stable states of the structure. Asymptotic results for the number of 0-stable states (fixed points) for the usual case $d = 1$ of linear threshold networks with Gaussian interactions have been reported in the literature (cf. [5]–[7]). We extend the technique used by McEliece and Posner [7] to obtain our general results.

As a function of n, let d_n explicitly represent the degree of the homogeneous threshold network, with the constraint $d_n < n - 1$. To avoid trivialities we restrict ourselves to $n \geq 3$. For any given n, and margin $\mathcal{B} > 0$, let $F_{(n, d_n, \mathcal{B})}$ denote the expected number of $\mathcal{B}$-stable states of a homogeneous network of degree d_n. In the following we will estimate $F_{(n, d_n, \mathcal{B})}$ under various assumptions on d_n and $\mathcal{B}$.

Let $P_{(n, d_n, \mathcal{B})}$ denote the probability that a given state $\boldsymbol{u}$ is $\mathcal{B}$-stable. Clearly, $F_{(n, d_n, \mathcal{B})} = 2^n P_{(n, d_n, \mathcal{B})}$. Without loss of generality, we assume that $\boldsymbol{u} = (1, 1, \cdots, 1)$. For each n, and $i = 1, \cdots, n$, define the sequence of random variables S_n^i by

$$S_n^i = \sum_{I \in \mathcal{I}_{d_n} : i \in I} w_I.$$

For $\boldsymbol{u}$ to be *$\mathcal{B}$-stable*, we require that $S_n^i > \mathcal{B}$ for $i = 1, \cdots, n$.

Now, for each n, the random variables S_n^i, $i = 1, \cdots, n$ are zero-mean, identically distributed, and jointly normal. Set

$$p_n = \binom{n-2}{d_n}^{1/2}, \qquad q_n = \binom{n-2}{d_n - 1}^{1/2}. \tag{7}$$

Then, we have

$$\boldsymbol{E}\left(S_n^i S_n^j\right) = \begin{cases} p_n^2 + q_n^2, & \text{if } i = j, \\ q_n^2, & \text{if } i \neq j. \end{cases}$$

Now, let $\{c_n\}$ be the sequence

$$c_n = \frac{n q_n^2}{p_n^2} = \frac{d_n}{1 - \left(\frac{d_n + 1}{n}\right)}, \tag{8}$$

and define the sequence of functions f_n by

$$f_n(t) = \log \Phi(t) - \frac{(t + \mathcal{B}/p_n)^2}{2c_n}, \tag{9}$$

where, in usual notation, $\Phi(t) = \int_{-\infty}^{t} \varphi(s)\, ds$ is the normal distribution function, and $\varphi(s) = (2\pi)^{-1/2} e^{-s^2/2}$ is the standard normal density function.

Proposition 2:

$$F_{(n, d_n, \mathcal{B})} = \frac{2^n \sqrt{n}}{\sqrt{2\pi c_n}} \int_{-\infty}^{\infty} e^{n f_n(t)}\, dt. \tag{10}$$

Proof: We use the principle of equivalent Gaussians. Let $X^0, X^1, \cdots, X^n$ be i.i.d., $\mathcal{N}(0, 1)$ random variables. Construct the random variables Y_n^i, $i = 1, \cdots, n$, by

$$Y_n^i = q_n X^0 + p_n X^i.$$

The random variables $\{Y_n^i\}_{i=1}^n$ are jointly normal, and have the same expectations and covariances as the random variables $\{S_n^i\}_{i=1}^n$. Hence,

$$\begin{aligned} P_{(n, d_n, \mathcal{B})} &= \boldsymbol{P}\left\{S_n^i > \mathcal{B}, i = 1, \cdots, n\right\} \\ &= \boldsymbol{P}\left\{Y_n^i > \mathcal{B}, i = 1, \cdots, n\right\} \\ &= \boldsymbol{P}\left\{X^i > -\frac{q_n}{p_n} X^0 + \frac{\mathcal{B}}{p_n}, i = 1, \cdots, n\right\} \\ &= \int_{-\infty}^{\infty} \boldsymbol{P}\left\{X^i > -\frac{q_n}{p_n} t + \frac{\mathcal{B}}{p_n}, i = 1, \cdots, n\right\} \varphi(t)\, dt \\ &= \int_{-\infty}^{\infty} \Phi\left(\frac{q_n}{p_n} t - \frac{\mathcal{B}}{p_n}\right)^n \varphi(t)\, dt. \end{aligned}$$

The result follows from the defining equations (8) and (9). □

We will estimate the expected number of $\mathcal{B}$-stable points given by (10) for large n using variants of Laplace's technique to estimate the integral (cf. de Bruijn [13], for instance). Rather careful asymptotic estimates are required, however, as the integral depends critically on the functions f_n, and these depend both on n and the interaction orders d_n.

It will be convenient to consider margins of the following form:

$$\mathcal{B} = \beta p_n c_n^{\alpha}, \qquad \alpha \geq 0, \qquad \beta \geq 0.$$

For given degrees of interaction d_n the expected number of $\mathcal{B}$-stable states will depend solely on the choices of the parameters $\alpha \geq 0$ and $\beta \geq 0$.

B. $\mathcal{B}$-Stability

Define the positive function

$$\psi(t) = \frac{\varphi(t)}{\Phi(t)} = \frac{e^{-t^2/2}}{\int_{-\infty}^{t} e^{-s^2/2}\, ds}.$$

It is easy to verify (see Lemma A.1, Appendix A) that, for every given degree d_n and margin $\mathcal{B}$, the function $f_n(t)$ has a unique maximum at $t = t_n$ where t_n satisfies

$$\psi(t_n) = \frac{t_n}{c_n} = \frac{\mathcal{B}}{p_n c_n}. \tag{11}$$

Note that t_n depends implicitly on both the margin $\mathcal{B}$ and the degree of interaction d_n. The following lemma, which we prove in Appendix B, provides the sought after estimate for $F_{(n, d_n, \mathcal{B})}$.

Lemma 1: Let $\mathcal{B} = \beta p_n c_n^{\alpha}$ with $0 \leq \alpha \leq 1$ and $\beta \geq 0$. If $d_n = o(n)$, then

$$F_{(n, d_n, \mathcal{B})} \sim \frac{2^n \left(1 + \frac{f_n(t_n)}{\log 2}\right)}{\sqrt{-c_n f_n''(t_n)}} \qquad (n \to \infty). \tag{12}$$

We are now in position to state the main theorem.

Theorem 1: Let $d_n = o(n)$, and consider margins of the form $\mathcal{B} = \beta p_n \sqrt{c_n}$, with $\beta \geq 0$. Then there are constants β_1 and β_2, with $0 < \beta_1 \leq \beta_2$, such that as $n \to \infty$,

a) $F_{(n, d_n, \mathcal{B})}$ increases exponentially with n whenever $0 \leq \beta < \beta_1$.

b) $F_{(n, d_n, \mathcal{B})}$ decreases exponentially with n whenever $\beta > \beta_2$.

Proof: We consider the cases, $\{d_n\}$ bounded and $\{d_n\}$ unbounded, separately.

Case 1: $\{d_n\}$ is bounded.

From (8) it is clear that $c_n \sim d_n$ is bounded. Consequently, from (11), (25), and (26) it follows that in (12) the term $f_n(t_n)$ is bounded while the term $\sqrt{-c_n f_n''(t_n)}$ is bounded away from zero. It is clear then that, for large n, the behavior of $F_{(n, d_n, \mathcal{B})}$ as β varies is determined entirely by the sign of the exponent in (12). Now, from (25), we have

$$f_n(t_n) = -\frac{t_n^2}{2}\left(1 + \frac{1}{d_n}\right) - \frac{\beta t_n}{\sqrt{d_n}} - \frac{\beta^2}{2} + \log\left(\frac{d_n}{\sqrt{2\pi}(t_n + \beta\sqrt{d_n})}\right) + O\left(\frac{1}{n}\right).$$

where t_n is bounded and satisfies (11). It is easy to verify that if $\beta = 0$ then $1 + f_n(t_n)/\log 2$ is positive and bounded away from zero, i.e., the expected number of fixed points (0–stable states) increases exponentially with n (see Table II for a listing of exponents for some fixed degrees of interaction). Now, for every n, $F_{(n, d_n \mathcal{B})}$ decreases monotonically as β increases (the expected number of $\mathcal{B}$-stable states is a monotonically decreasing function of the margin), and an examination of the previous asymptotic estimate for $f_n(t_n)$ shows that as β increases $f_n(t_n)$ eventually decreases sufficiently for $1 + f_n(t_n)/\log 2$ to become negative. Recalling that d_n takes values only in some finite set, by assumption, from the previous equation we can find $0 < \beta_1 \leq \beta_2$ such that

$$\limsup \frac{1}{\log 2}\left[-\frac{t_n^2}{2}\left(1 + \frac{1}{d_n}\right) - \frac{\beta_1 t_n}{\sqrt{d_n}} - \frac{\beta_1^2}{2} + \log\frac{d_n}{\sqrt{2\pi}(t_n + \beta_1\sqrt{d_n})}\right] = -1, \qquad (13)$$

$$\liminf \frac{1}{\log 2}\left[-\frac{t_n^2}{2}\left(1 + \frac{1}{d_n}\right) - \frac{\beta_2 t_n}{\sqrt{d_n}} - \frac{\beta_2^2}{2} + \log\frac{d_n}{\sqrt{2\pi}(t_n + \beta_2\sqrt{d_n})}\right] = -1. \qquad (14)$$

As $\sqrt{-c_n f_n''(t_n)}$ is bounded above and away from zero, it follows from (12) that for every $\beta < \beta_1$ there is $\epsilon(\beta) > 0$ such that $F_{(n, d_n, \mathcal{B})} = \Omega(2^{n\epsilon(\beta)})$; similarly, for every $\beta > \beta_2$ we can find $\delta(\beta) > 0$ such that $F_{(n, d_n, \mathcal{B})} = O(2^{-n\delta(\beta)})$.

Case 2: $d_n \to \infty$ such that $d_n = o(n)$.

For a choice of margin $\mathcal{B} = \beta p_n \sqrt{c_n}$, with $\beta > 0$, we have from (9) that

$$f_n(t_n) = \log \Phi(t_n) - \frac{(t_n + \beta\sqrt{c_n})^2}{2c_n} = -\sum_{k=1}^{\infty} \frac{\Phi(-t_n)^k}{k} - \frac{(t_n + \beta\sqrt{c_n})^2}{2c_n}. \qquad (15)$$

TABLE I
CRITICAL VALUES OF MARGIN, β^*, FOR VARIOUS CHOICES OF FIXED DEGREE, d

d	β^*
1	0.0690
2	0.1214
3	0.1557
4	0.1792
5	0.1960
10	0.2349
25	0.2476
50	0.2316
100	0.2023
1000	0.0959

Further, using (24) and the asymptotic form for the error function, we have

$$\Phi(-t_n) = \frac{\varphi(t_n)}{t_n}\left(1 - O\left(\frac{1}{t_n^2}\right)\right) = \frac{\beta}{\sqrt{c_n \log c_n}}\left(1 \pm O\left(\frac{1}{\log c_n}\right)\right).$$

Substituting from (15), (24), and (26) in (12), we then have

$$F_{(n, d_n, \mathcal{B})} = 2^{n\left\{1 - \frac{\beta^2}{2\log 2} - O\left(\frac{\sqrt{\log c_n}}{\sqrt{c_n}}\right) - O\left(\frac{\log c_n}{n}\right)\right\}}.$$

Setting $\beta_1 = \beta_2 = \sqrt{2\log 2}$ in the theorem, it is clear that exponentially increasing behavior attains when $0 < \beta < \sqrt{2\log 2}$, while, for $\beta > \sqrt{2\log 2}$, the expected number of $\mathcal{B}$-stable states decreases exponentially. To complete the proof, we need to show that $F_{(n, d_n, \mathcal{B})}$ increases exponentially with n when $\beta = 0$. But this follows immediately because the expected number of $\mathcal{B}$-stable states is a monotonically decreasing function of the margin. □

For $d_n = d =$ constant, and margin $\mathcal{B} = \beta p_n \sqrt{c_n}$, the critical quantities $\beta_1 = \beta_2 = \beta^*$ of (13) and (14) can be precisely calculated. The critical values β^* are listed in Table I for a range of fixed interaction orders. Note that the critical values β^* appear to increase a maximum around $d = 25$, and then decrease monotonically.

More explicit results can be deduced from Lemma 1. In the range where the expected number of $\mathcal{B}$-stable points increases exponentially, the multiplying coefficients and exponents can themselves be precisely calculated given the interaction orders d_n. Particular cases of importance result when $\{d_n\}$ converges to some $d > 0$, and in particular, the case $d_n = d =$ constant, and the case $d_n \to \infty$ monotonically.

Consider the case where $d_n = d =$ constant. Let $\alpha \geq 0$ and $\beta \geq 0$ specify the margin $\mathcal{B}$, and let s be the unique solution of the equation

$$\psi(s) = \frac{s + \beta d^\alpha}{d}.$$

The location t_n of the maximum of f_n (satisfying (11)) can be approximated up to terms of the order of n^{-2} by

$$t_n = s + \frac{\kappa}{n} + O\left(\frac{1}{n^2}\right),$$

where κ is independent of n and satisfies

$$\kappa = \frac{d(d+1)[s + (1-\alpha)\beta d^\alpha]}{d + (s + \beta d^\alpha)[s(d+1) + \beta d^\alpha]}. \qquad (16)$$

Using this approximation for t_n in Lemma 1 and collecting all terms up to the order of n^{-2} in the exponent in (12) yields the following result.

TABLE II
THE BEHAVIOR OF THE EXPECTED NUMBER OF FIXED POINTS, $F_{(n,d,0)} \sim k_d 2^{nw_d}$, FOR DIFFERENT VALUES OF FIXED DEGREE OF INTERACTION, d.

d	k_d	w_d
1	1.0505	0.2874
2	1.1320	0.4265
3	1.2178	0.5124
4	1.3031	0.5721
5	1.3868	0.6165
10	1.7784	0.7382
25	2.7867	0.8541
50	4.2207	0.9104
100	6.7176	0.9466
1000	39.3421	0.9917

Corollary 1: If $d_n = d > 0$ and $\mathcal{B} = \beta p_n c_n^\alpha$ with $\alpha \geq 0$ and $\beta \geq 0$, then, as $n \to \infty$, $F_{(n,d_n,\mathcal{B})} \sim k_d 2^{nw_d}$, where the multiplying coefficient k_d and exponent w_d are independent of n (and depend solely on the interaction order d and the margin parameters α and β); specifically,

$$w_d = 1 - \frac{d+1}{2d\log 2}\left(s^2 + \frac{2s\beta d^\alpha}{d+1} + \frac{\beta^2 d^{2\alpha}}{d+1}\right) - \frac{1}{\log 2}\log\left(\frac{\sqrt{2\pi}(s+\beta d^\alpha)}{d}\right),$$

and k_d can be expressed in the form Ce^D where

$$C = \left[\frac{d}{s^2(d+1) + s\beta d^\alpha(d+2) + \beta^2 d^{2\alpha} + d}\right]^{1/2},$$

and with κ as in (16),

$$D = \frac{d+1}{2d}\left[s^2 - 2s\{\kappa - \beta d^\alpha(1-\alpha)\} + \beta^2 d^{2\alpha}(1-2\alpha) + 2d\right] - \kappa\beta d^{\alpha-1} - \frac{\alpha\beta d^a(d+1)}{s+\beta d^a} - \frac{\kappa}{s+\beta d^\alpha}.$$

An important special case results when we choose the margin to be identically zero.

Corollary 2: If $d_n = d > 0$, the expected number of fixed points is asymptotically $\sim k_d 2^{nw_d}$, where

$$w_d = 1 - \frac{1}{\log 2}\left[\frac{(d+1)s^2}{2d} + \log\left(\frac{s\sqrt{2\pi}}{d}\right)\right],$$

$$k_d = \sqrt{\frac{d}{d + s^2(d+1)}}\exp\left(\frac{(d+1)s^2}{2d}\right).$$

Table II lists the exponent w_d and the multiplying coefficient k_d for various choices of fixed interaction order d with a choice of zero margin.

The monotonicity of $F_{(n,d_n,\mathcal{B})}$ with $\mathcal{B}$ yields

Corollary 3: Let $\mathcal{B} = \beta p_n c_n^\alpha$ be the margin. If $d_n \to \infty$ such that $d_n = o(n)$ then

a) the expected number of $\mathcal{B}$-stable states increases exponentially with n if $0 \leq \alpha < 1/2$ and $\beta \geq 0$;
b) the expected number of $\mathcal{B}$-stable states asymptotically tends to zero as $n \to \infty$ if $\alpha > 1/2$ and $\beta > 0$.

Note from Table II that as d becomes large $F_{(n,d_n,\mathcal{B})}$ approaches 2^n. This is supported by the following result that gives the number of fixed points (0-stable states) when the interaction orders are allowed to grow large.

Corollary 4: If as $n \to \infty$, for any fixed choice of τ with $0 < \tau < 1$, $\{d_n\}$ satisfies $d_n = \Omega[n/(\log n)^\tau]$, and $d_n = o(n)$, then the expected number of fixed points (zero margin) is given by $F_{(n,d_n,0)} \sim k_{d_n} 2^{nw_{d_n}}$, as $n \to \infty$, where

$$k_{d_n} = \frac{d_n}{2\sqrt{2\pi}\log d_n},$$

$$w_{d_n} = 1 - \frac{\log d_n}{d_n \log 2} + \frac{\log\log d_n}{2d_n\log 2} + \frac{\log\left(\sqrt{4\pi}/e\right)}{d_n\log 2}.$$

Proof: Consider the exponent in the integrand of (10). We have

$$nf_n(t_n) = n\log\Phi(t_n) - \frac{nt_n^2}{2c_n} = -\sum_{k=1}^{\infty}\frac{n}{k}\Phi(-t_n)^k - \frac{nt_n^2}{2c_n}.$$

Using the asymptotic formula for the tails of the normal distribution (cf. Feller's text [14], for instance) together with Lemma A.3 (23) and (8) and (26) in (12) completes the proof. □

Note that for this case, the multiplying coefficient k_{d_n} and exponent w_{d_n} assume particularly simple closed form expressions depending solely on the interaction order d_n. Note also that $\omega_{d_n} \to 1$ as $n \to \infty$, as is expected. The growth of d_n with n is rather rapid in Corollary 4. Results akin to Corollary 4 can be computed for slower rates of growth of d_n (for instance, $d_n = n^\alpha$, $0 < \alpha < 1$). We do not yet have rigorous results, however, for the case where d_n scales linearly with n.

IV. NONHOMOGENEOUS NETWORKS

A. Higher Order Spin Glasses

The nonhomogeneous case has several more degrees of interconnection freedom. The results of the last section can, however, be simply extended to this case.

Analogously with (7) and (8), let

$$\hat{p}_n = \left[\sum_{k=1}^{d_n}\binom{n-2}{k}\right]^{1/2}$$

and

$$\hat{c}_n = \frac{n\sum_{j=1}^{d_n}\binom{n-2}{j-1}}{\sum_{j=1}^{d_n}\binom{n-2}{j}},$$

and to every choice of margin $\mathcal{B} = \beta p_n c_n^\alpha$ (fixed $\beta \geq 0$ and $\alpha \geq 0$) in the homogeneous case associate a margin $\hat{\mathcal{B}} = \beta\hat{p}_n\hat{c}_n^\alpha$ in the nonhomogeneous case. Define the sequence of functions $\hat{f}_n$ (corresponding to (9)) by

$$\hat{f}_n(t) = \log\Phi(t) - \frac{\left(t + \hat{\mathcal{B}}/\hat{p}_n\right)^2}{2\hat{c}_n}. \tag{17}$$

Let $\hat{F}_{(n,d_n,\hat{\mathcal{B}})}$ denote the expected number of $\hat{\mathcal{B}}$-stable states of a nonhomogeneous algebraic threshold network of degree d_n with Gaussian interactions.

Proposition 3:

$$\hat{F}_{(n,d_n,\hat{\mathcal{B}})} = \frac{\sqrt{n}2^n}{\sqrt{2\pi\hat{c}_n}}\int_{-\infty}^{\infty} e^{n\hat{f}_n(t)}dt. \tag{18}$$

Proof: For $i = 1, \cdots, n$, set

$$\hat{S}_n^i = \sum_{j=1}^{d_n}\sum_{I\in\mathcal{I}_j : i\in I} w_I.$$

Noting that $\hat{F}_{(n,d_n,\hat{\mathcal{B}})} = 2^n \boldsymbol{P}\left\{\hat{S}_n^i > \hat{\mathcal{B}},\ i = 1, \cdots, n\right\}$, the proof follows the same outline as that for Proposition 2. □

Theorem 2: If $d_n = o(n)$ then $\hat{F}_{(n,d_n,\hat{\mathcal{B}})} \sim F_{(n,d_n,\mathcal{B})}$ as $n \to \infty$.

Proof: We use the following inequality due to Blake and Darabian [15]. Set $r = d/(n-d+1)$. Then,

$$\frac{1}{1-r}\left(1 - \frac{(n+1)r^2}{d(n-d+1)(1-r)^2}\right) < \frac{\sum_{j=0}^{d}\binom{n}{j}}{\binom{n}{d}} < \frac{1}{1-r}.$$

For $d_n = o(n)$, we hence have

$$\hat{c}_n = \frac{n\sum_{j=1}^{d_n}\binom{n-2}{j-1}}{\sum_{j=1}^{d_n}\binom{n-2}{j}} = \frac{n\binom{n-2}{d_n-1}}{\binom{n-2}{d_n}}\left(1 - \frac{1}{n} - O\left(\frac{d_n}{n^2}\right)\right)$$
$$= c_n\left(1 - \frac{1}{n} - O\left(\frac{d_n}{n^2}\right)\right).$$

The analysis in Theorem 1 now continues to hold *in toto.* □

So far, we have considered relatively small interaction orders, $d_n = o(n)$. A theoretically important case results when d_n is allowed to grow linearly with n. In fact, as d_n approaches n, almost all dichotomies of 2^n points in binary n-space can be separated by a nonhomogeneous network (Venkatesh and Baldi [2]). It is useful, hence, to estimate the number of fixed points $\hat{F}_{(n,d_n,0)}$ for the random case when d_n grows linearly with n.

Theorem 3: If $d_n \sim n/2$ then $\hat{F}_{(n,d_n,0)} \sim 2^n/(n+1)$ as $n \to \infty$.

Proof: If $d_n \sim n/2$ then $\hat{c}_n = n[1 \pm O(1/\sqrt{n})]$. Hence, from (17) and (18),

$$\hat{F}_{(n,d_n,\hat{\mathcal{B}})} = \frac{2^n}{\sqrt{2\pi}}\left(1 \pm O\left(\frac{1}{\sqrt{n}}\right)\right)\int_{-\infty}^{\infty}\Phi(t)^n e^{-t^2/2}e^{-t^2/v_n}\,dt, \tag{19}$$

where $v_n = \Omega(\sqrt{n})$. Fix $0 < \tau < 1/4$. Then,

$$\frac{\left[\Phi(n^\tau)^{n+1} - \Phi(-n^\tau)^{n+1}\right]e^{-n^{2\tau}/v_n}}{n+1} < \frac{1}{\sqrt{2\pi}}\int_{-n^\tau}^{n^\tau}\Phi(t)^n e^{-t^2/2}e^{-t^2/v_n}\,dt < \frac{\Phi(n^\tau)^{n+1} - \Phi(-n^\tau)^{n+1}}{n+1},$$

while

$$0 \le \frac{1}{\sqrt{2\pi}}\int_{|t|\ge n^\tau}\Phi(t)^n e^{-t^2/2}e^{-t^2/v_n}\,dt \le e^{-n^{2\tau}/v_n} \cdot\left(\frac{1 - \Phi(n^\tau)^{n+1} + \Phi(-n^\tau)^{n+1}}{n+1}\right).$$

Now $\Phi(n^\tau)^{n+1} \to 1$, $\Phi(-n^\tau)^{n+1} \to 0$, and $n^{2\tau}/v_n \to 0$ as $n \to \infty$. Hence,

$$\frac{1}{\sqrt{2\pi}}\int_{-n^\tau}^{n^\tau}\Phi(t)^n e^{-t^2/2}e^{-t^2/v_n}\,dt \sim \frac{1}{n+1} \qquad (n \to \infty),$$
$$\frac{1}{\sqrt{2\pi}}\int_{|t|\ge n^\tau}\Phi(t)^n e^{-t^2/2}e^{-t^2/v_n}\,dt = o\left(\frac{1}{n+1}\right) \qquad (n \to \infty).$$

The proof is completed by substitution in (19). □

B. Random Energy Model

The dynamics of the symmetric interaction systems previously considered are characterized by Hamiltonians or energies. The determination of the number of fixed points of such a system is hence, equivalent to counting the number of states which form (local) energy minima. For higher order spin glasses, the energy of each state given by (3) is an $\mathcal{N}\left(0, \binom{n-1}{d_n-1}\right)$ random variable. The energies of different states are *dependent,* identically distributed normal random variables.

The random energy model (cf. Derrida [11]) is an allied system which *assigns* energies as i.i.d., $\mathcal{N}(0, 1)$ random variables to the vertices of the hypercube. State energies are now independent normal random variables. Such an assignment of state energies results in random acyclic orientations of the vertices of the hypercube (cf. Baldi [16]) defined by state transitions: $\boldsymbol{u} \mapsto \boldsymbol{v}$ iff $H(\boldsymbol{u}) > H(\boldsymbol{v})$. For any given assignment of energies, the corresponding acyclic orientation can be realized by a (nonhomogeneous) threshold network with degree $d_n = n$. In particular, we have 2^n interaction coefficients for such a system so that any particular assignment of 2^n state energies can be realized for a particular choice of coefficients.

Let G_n be the number of local energy minima corresponding to a random acyclic orientation.

Theorem 4:

$$\boldsymbol{E}(G_n) = \frac{2^n}{n+1},$$
$$\operatorname{var}(G_n) = \frac{(n-1)2^{n-1}}{(n+1)^2}.$$

Proof: Let $\boldsymbol{s}^i$, $i = 1, \cdots, 2^n$, enumerate the vertices of the hypercube. For $i = 1, \cdots, 2^n$, let the random variable I^i be the indicator for state $\boldsymbol{s}^i$, i.e.,

$$I^i = \begin{cases} 1, & \text{if } s^i \text{ is an energy minimum,} \\ 0, & \text{otherwise.} \end{cases}$$

We then have $G_n = \sum_{i=1}^{2^n} I^i$. The probability $p = \boldsymbol{P}\{I^i = 1\}$ that any particular state is a local minimum is just the probability that it is assigned a lower energy value than any of its n nearest neighbors (at Hamming distance one from it). As the assigned energies are i.i.d. random variables, we have $p = 1/(n+1)$. Hence, the expected number of fixed points is $2^n/(n+1)$. We now compute the joint probability that two states $\boldsymbol{s}^i$ and $\boldsymbol{s}^j$ are energy minima. Let d_{ij} represent the Hamming distance between $\boldsymbol{s}^i$ and $\boldsymbol{s}^j$. It is easy to see that

$$\boldsymbol{P}\{I^i I^j = 1\} = \begin{cases} 1/(n+1)^2, & \text{if } d_{ij} > 2, \\ 1/n(n+1)^2, & \text{if } d_{ij} = 2, \\ 0, & \text{if } d_{ij} = 1, \\ 1/(n+1), & \text{if } d_{ij} = 0. \end{cases}$$

Now, we have

$$\begin{aligned}\operatorname{var}(G_n) &= \boldsymbol{E}(G_n^2) - (\boldsymbol{E}G_n)^2 \\ &= \sum_{i=1}^{2^n}\sum_{j=1}^{2^n}\boldsymbol{P}\left\{I^i I^j = 1\right\} - \left(\sum_{i=1}^{2^n}\boldsymbol{P}\left\{I^i = 1\right\}\right)^2 \\ &= \frac{2^n}{n+1} + 2\sum_{\substack{i<j \\ d_{ij}=2}}\frac{1}{n(n+1)} \\ &\quad + 2\sum_{\substack{i<j \\ d_{ij}>2}}\frac{1}{(n+1)^2} - \left(\frac{2^n}{n+1}\right)^2.\end{aligned}$$

Collecting terms and simplifying yields the final result. □

Note that the result of Theorem 4 provides anecdotal support for the result of Theorem 3 as a sort of limiting result. Stronger results can be shown for the random energy model: the number of fixed points, G_n, exhibits central tendency. Let G_n^* denote the normalized r.v.

$$G_n^* = \frac{G_n - \boldsymbol{E}G_n}{\sqrt{\operatorname{var} G_n}}.$$

Theorem 5: There is an absolute positive constant C such that for every x

$$|\boldsymbol{P}(G_n^* < x) - \Phi(x)| \le C2^{-n/4}.$$

We refer the reader to the papers by Baldi *et al.* [17], [18] for a proof of the theorem. It is an open question whether the number of fixed points of higher order spin glasses also exhibits central tendency.

V. Conclusion

We have rigorously estimated the expected number of stable points of higher order spin glasses with generalized Gaussian interactions. The critical feature observed here is the threshold phenomenon that is evidenced in the expected number of fixed points around a range of degree dependent critical margins. For margins below the critical range we have shown a precise exponentially increasing form of the solution, while for margins greater than the critical range we have shown that the expected number of fixed points decreases exponentially with the number of nodes in the network. Open questions remain on a more precise determination (than the mean) of the number of fixed points (as a function of the margin), and in particular, on whether there is central tendency as in the random energy model.

The results of this correspondence appear to have relevance to the programmed situation where interaction stengths are to be chosen for which *specified* collections of binary n-tuples are fixed points with some desired radius of attraction. In such cases, it is important to be cognisant of the number of extraneous fixed points—and their radii of attraction—that are developed incidentally. Rigorous results have, however, been shown only of the linear interaction case ($d = 1$) with interactions programmed by the outer product algorithm (Komlós and Paturi [9]). The analysis appears to be substantially harder for higher order cases, even for the relatively simple outer product algorithm (cf. Newman's earlier paper [4] and our two concurrent papers [2], [3] for illustrations of the difficulties caused by the more severe statistical dependences in higher order cases). The extraneous fixed point structure of other algorithms, such as the spectral algorithm (Venkatesh and Psaltis [10]), is even less understood, especially in the higher order versions. It is not readily apparent whether the results derived here for the case of random, symmetric interactions (especially Theorem 3 and the corollaries) can be utilized in a rigorous analysis in programmed cases; nonetheless, these results may provide qualitative indications of behaviors that may be expected in programmed cases, especially when the dependences are weak.

Appendix A
Properties of f_n

Lemma A. 1: For each n (and any choice of margin $\mathcal{B} > 0$ and degree d_n),

a) f_n is a convex ∩, strictly negative function with a unique maximum at $t = t_n$;
b) for $t \ge 1$, f_n'' increases monotonically to $-1/c_n$ as $t \to \infty$.

Proof: First we claim that ψ is a positive, monotone decreasing function. Clearly $\psi(t) > 0$ for all t. Consider $\psi'(t) = -t\psi(t) - \psi(t)^2$. We have $\psi'(t) < 0$ for $t \ge 0$. Now, for $t > 0$, consider

$$\psi(-t) = \frac{\varphi(-t)}{\Phi(-t)} > \frac{\varphi(-t)}{\frac{1}{|t|}\varphi(-t)} = t > 0.$$

Hence, $\psi'(t) < 0$ for all t so that ψ is monotone decreasing. By repeated differentiation of (9), we have

$$f_n'(t) = \psi(t) - \frac{t + \mathcal{B}/p_n}{c_n}, \tag{20}$$

$$f_n''(t) = -\psi(t)^2 - t\psi(t) - \frac{1}{c_n}. \tag{21}$$

Now $f_n''(t) = \psi'(t) - 1/c_n < 0$ for all t so that f_n is strictly convex ∩, while the monotonicity of ψ guarantees a unique solution at $t = t_n$ to $f_n'(t) = 0$. As $f_n(t) < 0$ for all t by inspection of (9), part a) follows. Now, note that

$$[t\psi(t)]' = -\psi(t)[t^2 + t\psi(t) - 1] \le -t\psi(t)^2 < 0 \qquad (t \ge 1).$$

Hence both $\psi(t)$ and $t\psi(t)$ decrease monotonically to zero so that b) follows. □

Lemma A.2: For each n, f_n has derivatives of all orders, and in fact, for $k \ge 3$, the derivatives $f_n^{(k)}$ are independent of n and have the representation

$$f_n^{(k)}(t) = \sum_{l=0}^{[k/2]} \sum_{m=1}^{k-2l} c_{l,m}^{(k)} t^{k-2l-m} \psi(t)^m, \tag{22}$$

where the coefficients $c_{l,m}^{(k)}$ are real constants independent of n, and $c_{0,1}^{(k)} = (-1)^{k-1}$.

Proof: Note that for $k \ge 3$ we have $f_n^{(k)}(t) = \frac{d^{k-1}\psi(t)}{dt^{k-1}}$. The result follows by induction. □

Lemma A.3: Let $\mathcal{B} = \beta p_n c_n^\alpha$ with $\alpha \ge 0$ and $\beta \ge 0$, and let f_n achieve its maximum at t_n. Then, as $n \to \infty$:

a) if $d_n \to d$, then $t_n \to s$ where s satisfies

$$\psi(s) - \frac{s}{d} = \beta d^{\alpha - 1};$$

b) if $d_n \to \infty$, and $\alpha = 0$ or $\beta = 0$, then

$$t_n = \left[2\log c_n - \log\log c_n - \log(4\pi) + O\left(\frac{1}{\sqrt{\log c_n}}\right)\right]^{1/2}; \tag{23}$$

c) if $d_n \to \infty$, $0 < \alpha < 1$, and $\beta > 0$, then

$$t_n = \left[2(1-\alpha)\log c_n - 2\log\left(\beta\sqrt{2\pi}\right) - O\left(\frac{\sqrt{\log c_n}}{c_n^\alpha}\right)\right]^{1/2}. \tag{24}$$

Proof: Part a) follows by continuity of ψ as $c_n \to d_n$ ($n \to \infty$) from (8). Parts b) and c) can be verified by direct substitution.

Remarks:

$$f_n(t_n) = -\frac{t_n^2}{2}\left(1 + \frac{1}{c_n}\right) - \beta t_n c_n^{\alpha-1} - \frac{\beta^2}{2} c_n^{2\alpha-1} - \log\left(\frac{\sqrt{2\pi}}{c_n}(t_n + \beta c_n^\alpha)\right), \tag{25}$$

$$f_n'(t_n) = 0,$$

$$f_n''(t_n) = -\frac{t_n^2}{c_n}\left(1 + \frac{1}{c_n}\right) - t_n\beta c_n^{\alpha-1}\left(1 + \frac{2}{c_n}\right) - \beta^2 c_n^{2\alpha-2} - \frac{1}{c_n}, \tag{26}$$

$$f_n^{(k)}(t_n) \sim (-1)^{k-1} \frac{t_n^{k-1}}{c_n}(t_n + \beta c_n^\alpha), \quad k \geq 3, \quad d_n \to \infty. \tag{27}$$

Note that for $k \geq 3$, $f_n^{(k)}(t)$ *does not depend on* n *any more* so that uniform bounds can be obtained.

We will seek to approximate the functions, $f_n(t)$, by the first few terms of a Taylor series expansion. Specifically, for particular choices of $\epsilon_n > 0$ and $\delta_n > 0$, we use

$$|t - t_n| < \delta_n \Rightarrow \left| f_n(t) - f_n(t_n) - f_n''(t_n)\frac{(t-t_n)^2}{2} \right| < \epsilon_n \frac{(t-t_n)^2}{2}. \tag{28}$$

The next two lemmas outline conditions under which this holds.

Lemma A.4: If the sequence $\{d_n\}$ is bounded then for any specification of margin $\mathcal{B} = \beta p_n c_n^\alpha$ with $\alpha \geq 0$ and $\beta \geq 0$, and for any $\epsilon > 0$, we can find $\delta > 0$ uniform with respect to n such that (28) holds with a choice of $\epsilon_n = \epsilon$ and $\delta_n = \delta$.

Proof: Set $g_n(t) = f_n(t) - f_n(t_n) - f_n''(t_n)(t-t_n)^2/2$.

We have $g_n(t_n) = g_n'(t_n) = g_n''(t_n) = 0$. Applying the mean value theorem, we can find $0 < \alpha < 1$ such that $g_n(t_n + \alpha\zeta) = g_n'(t_n + \alpha\zeta)$, while $g_n'(t_n + \alpha\zeta) = o(\alpha\zeta)$ as $\zeta \to 0$. Hence, $g_n(t_n + \zeta) = o(\zeta^2)(\zeta \to 0)$. Thus, for each n, and every $\epsilon > 0$, we can find $\delta_n > 0$ such that $|g_n(t)| < \epsilon(t-t_n)^2/2$ whenever $|t - t_n| < \delta_n$.

Now let d_n takes value in the finite set $\left\{\mu^{(1)}, \cdots, \mu^{(K)}\right\}$, without loss of generality. For $i = 1, \cdots, K$, set

$$f^i(t) = \log\Phi(t) - \frac{\left(t + \beta\left(\mu^{(i)}\right)^\alpha\right)^2}{2\mu^{(i)}},$$

$$g^i(t) = f^i(t) - f^i\left(t^{(i)}\right) - f^{i''}\left(t^{(i)}\right)\frac{\left(t - t^{(i)}\right)^2}{2},$$

where f^i has its maximum at $t^{(i)}$. Then, for every $\epsilon > 0$, there exists $\delta^{(i)} > 0$ such that $|g^i(t)| < \epsilon\left(t - t^{(i)}\right)^2/4$ whenever $|t - t^{(i)}| < \delta^{(i)}$. Now $c_n = d_n + O(1/n)$ so that from (11) it follows that $t_n = t^{(i)} + O(1/n)$ for some $i \in \{1, \cdots, K\}$. [As ψ is monotone decreasing, we have:

$$t_n - t^{(i)} < x = \left(\frac{t^{(i)}}{d_n} - \frac{t^{(i)}}{c_n}\right) \Big/ \left(\frac{1}{c_n}\right) = t^{(i)} O\left(\frac{1}{n}\right).$$

As d_n is bounded we are guaranteed that $t^{(i)}$ is also bounded, so that the result follows.] From (9), (25), and (26) it hence, follows that

$$|g_n(t)| = |g^i(t)| + O\left(\frac{1}{n}\right) < \epsilon\frac{(t-t_n)^2}{4} + O\left(\frac{1}{n}\right) \qquad \left(|t - t_n| < \delta^{(i)}\right).$$

We can, hence, choose N such that for $n \geq N$, we have $|g_n(t)| < \epsilon(t-t_n)^2/2$ whenever $|t - t_n| < \min\left\{\delta^{(1)}, \cdots, \delta^{(K)}\right\}$. We can finally choose a smallest $\delta = \min\left\{\delta_1, \cdots, \delta_N, \delta^{(1)}, \cdots, \delta^{(K)}\right\}$ to establish uniformity. □

Lemma A.5: Let $\mathcal{B} = \beta p_n c_n^\alpha$ be the margin. If $d_n \to \infty$ as $n \to \infty$, then (28) holds for n large enough for the following choices of ϵ_n and δ_n:

a) $\epsilon_n = \lambda \log c_n / c_n$ and $\delta_n = \lambda/\sqrt{128 \log c_n}$, if $\alpha = 0$ or $\beta = 0$.
b) $\epsilon_n = \lambda\sqrt{\log c_n}/c_n^{1-\alpha}$ and $\delta_n = \lambda/(8\beta(1-\alpha)\sqrt{\log c_n})$, if $0 < \alpha < 1$ and $\beta > 0$.

Here, $\lambda > 0$ is a suitably small (but fixed) choice of parameter.

Proof: We will prove the result for the case $\{\alpha = 0 \text{ or } \beta = 0\}$; the proof for the case $\{0 < \alpha < 1 \text{ and } \beta > 0\}$ is similar.

Consider a choice of margin $\mathcal{B} = \beta p_n c_n^\alpha$ with $\alpha = 0$ or $\beta = 0$. Set $\epsilon_n = \lambda \log c_n / c_n$ and $\delta_n = \lambda/\sqrt{128 \log c_n}$ for some $\lambda > 0$ to be specified suitably small. In the proof of Lemma A.4, set $\epsilon = \epsilon_n$. Now, it suffices to show that $|g_n'(t_n + \zeta)| < \epsilon_n|\zeta|/2$ whenever $|\zeta| < \delta_n = \lambda/\sqrt{128 \log c_n}$, $(n \to \infty)$. We have

$$|g_n'(t_n + \zeta)| = |f_n'(t_n + \zeta) - f_n''(t_n)\zeta|.$$

By the mean value theorem, there exists $0 < \beta < 1$ such that

$$|f_n'(t_n + \zeta)| = |\zeta||f_n''(t_n + \beta\zeta)|.$$

Now, consider

$$|f_n'(t_n - \delta_n) + f_n''(t_n)\delta_n| = \delta_n| - f_n''(t_n - \beta\delta_n) + f_n''(t_n)| \leq \delta_n| - f_n''(t_n - \delta_n) + f_n''(t_n)|.$$

The last inequality follows from Lemma A.1 b) as f_n'' is negative and increases monotonically to $-1/c_n$ for large t, and by Lemma A.3 b) which ensures that $t_n \sim \sqrt{2\log d_n} \to \infty\ (n \to \infty)$. Using (21) with Lemma A.3 b), as $n \to \infty$ we have

$$\begin{aligned}|f_n'(t_n - \delta_n) + f_n''(t_n)\delta_n| &\leq \delta_n|\psi(t_n - \delta_n)^2 + (t_n - \delta_n)\psi(t_n - \delta_n) - \psi(t_n)^2 - t_n\psi(t_n)| \\ &\leq \frac{\delta_n t_n}{\sqrt{2\pi}} e^{-t_n^2/2}|e^{t_n\delta_n - \delta_n^2} - 1| + O\left(\frac{1}{c_n\sqrt{\log c_n}}\right).\end{aligned}$$

We have $t_n\delta_n = O(\lambda)$ so that for λ sufficiently small

$$|f_n'(t_n - \delta_n) + f_n''(t_n)\delta_n| \lesssim \delta_n\left[\frac{\sqrt{2}\delta_n t_n^2}{\sqrt{\pi}} e^{-t_n^2/2}\right] \sim \epsilon_n \frac{\delta_n}{2} \qquad (n \to \infty).$$

Similarly,

$$|f_n'(t_n - \delta_n) - f''(t_n)\delta_n| \lesssim \epsilon_n\frac{\delta_n}{2} \qquad (n \to \infty).$$

By Lemma A.1, these inequalities hold in the δ_n-neighborhood of t_n, and this completes the proof. □

Appendix B
The Main Lemma

We prove here Lemma 1, restated now for convenience.

Lemma 1: Let $\mathcal{B} = \beta p_n c_n^\alpha$ with $0 \leq \alpha \leq 1$ and $\beta \geq 0$. If $d_n = o(n)$, then

$$F_{(n, d_n, \mathcal{B})} \sim \frac{2^n\left(1 + \frac{f_n(t_n)}{\log 2}\right)}{\sqrt{-c_n f_n''(t_n)}} \qquad (n \to \infty).$$

Proof: We will consider separately the cases where $\{d_n\}$ is bounded and $\{d_n\}$ is unbounded.

Case 1: $\{d_n\}$ is bounded.

The sequence $f_n''(t_n)$ is bounded strictly away from both zero and infinity. Hence, set $\xi = \inf|f_n''(t_n)| > 0$ and $\kappa = \sup|f_n''(t_n)| < \infty$. Fix ϵ arbitrarily in the open interval $(0, \xi)$. Choose $\delta > 0$ uniform with respect to n by Lemma A.4. By Proposition 2, we have

$$\frac{\sqrt{2\pi c_n}}{2^n\sqrt{n}} F_{(n, d_n, \mathcal{B})} = \int_{|t-t_n|<\delta} e^{nf_n(t)}\,dt + \int_{|t-t_n|\geq\delta} e^{nf_n(t)}\,dt.$$

Let ϑ be a parameter, $|\vartheta| \le \epsilon$. Consider

$$\int_{t_n-\delta}^{t_n+\delta} e^{n(f_n''(t_n)+\vartheta)(t-t_n)^2/2}\,dt$$

$$= \left[\frac{-2\pi}{n(f_n''(t_n)+\vartheta)}\right]^{1/2} - 2\Phi\left(-\delta\sqrt{-n(f_n''(t_n)+\vartheta)}\right) \quad (29)$$

$$= \left[\frac{-2\pi}{n(f_n''(t_n)+\vartheta)}\right]^{1/2} - O\left(\frac{e^{-\delta^2 n(\xi-\vartheta)/2}}{\delta\sqrt{n(\xi-\vartheta)}}\right).$$

By Lemma A.4, it then follows that

$$\left[\frac{-2\pi}{n(f_n''(t_n)-\epsilon)}\right]^{1/2} - O\left(\frac{e^{-q(\delta)n}}{\sqrt{n}}\right) < e^{-nf_n(t_n)}\int_{t_n-\delta}^{t_n+\delta} e^{nf_n(t)}\,dt < \left[\frac{-2\pi}{n(f_n''(t_n)+\epsilon)}\right]^{1/2} - O\left(\frac{e^{-q(\delta)n}}{\sqrt{n}}\right),$$

where $q(\delta) > 0$ depends solely on δ. As ϵ was arbitrary, we have

$$\int_{t_n-\delta}^{t_n+\delta} e^{nf_n(t)}\,dt \sim e^{nf_n(t_n)}\left(\frac{-2\pi}{nf_n''(t_n)}\right)^{1/2} \qquad (n\to\infty). \quad (30)$$

Let $\zeta = \sup|f_n(t_n)|$. The sequence $\{f_n(t_n)\}$ is bounded so that $\zeta < \infty$. For each n, set

$$h_n(\delta) = \max\{f_n(t_n-\delta) - f_n(t_n),\ f_n(t_n+\delta) - f_n(t_n)\} < 0.$$

Then,

$$\int_{|t-t_n|\ge\delta} e^{nf_n(t)}\,dt = e^{nf_n(t_n)} \int_{|t-t_n|\ge\delta} e^{n(f_n(t)-f_n(t_n))}\,dt \le e^{nf_n(t_n)}\left\{e^{(n-1)h_n(\delta)-f_n(t_n)} \int_{|t-t_n|\ge\delta} \Phi(t)e^{-t^2/2c_n}\,dt\right\}. \quad (31)$$

Now let $h(\delta) = \sup_n h_n(\delta)$. As $\{d_n\}$ is bounded, Lemma A.1 ensures that $h(\delta) < 0$ strictly. Furthermore, for each n, $0 > h_n(\delta) \ge f_n(t) - f_n(t_n)$ whenever $|t-t_n| \ge \delta$, by Lemma A.1. Hence,

$$\int_{|t-t_n|\ge\delta} e^{nf_n(t)}\,dt \le e^{nf_n(t_n)}\left\{\sqrt{2\pi c_n e^{(n-1)h(\delta)+\zeta}}\right\}.$$

Hence, there exists $\gamma(\delta),\ p(\delta) > 0$ such that

$$\frac{\int_{|t-t_n|\ge\delta} e^{nf_n(t)}\,dt}{\int_{|t-t_n|<\delta} e^{nf_n(t)}\,dt} \lesssim \gamma(\delta)\sqrt{\kappa n c_n}\,e^{-p(\delta)n} \to 0 \qquad (n\to\infty),$$

so that (12) follows.

Case 2: $d_n \to \infty$ such that $d_n = o(n)$ as $n \to \infty$.

We prove the result for a choice of margin with $\alpha = 0$ or $\beta = 0$. Fix $\lambda > 0$ and choose $\epsilon_n = \lambda \log c_n / c_n$, and $\delta_n = \lambda/\sqrt{128 \log c_n}$. (Note that $c_n \sim d_n$ from equation (8).) Now, from (23) and (26), for $|\vartheta| \le \epsilon_n$ and for small λ, as $n \to \infty$,

$$-\left(f_n''(t_n) + \vartheta\right) = \frac{2\log c_n}{c_n}[1 \pm O(\lambda)].$$

As $n/c_n \sim n/d_n \to \infty$ $(n\to\infty)$, the first term on the right-hand side of (29) dominates the second, so that for a sufficiently small choice of λ, (30) continues to hold:

$$\int_{t_n-\delta_n}^{t_n+\delta_n} e^{nf_n(t)}\,dt \sim e^{nf_n(t_n)}\left(\frac{-2\pi}{nf_n''(t_n)}\right)^{1/2} \qquad (n\to\infty). \quad (32)$$

Now by Taylor's formula, we have

$$f_n(t_n \pm \delta_n) - f_n(t_n) = \frac{f_n''(t_n)\delta_n^2}{2} + \frac{1}{2}\int_{t_n}^{t_n\pm\delta_n} (t-t_n)^2 f_n'''(t)\,dt.$$

By Lemma A.2 and (23) and (27), we have

$$\left|\frac{1}{2}\int_{t_n}^{t_n\pm\delta_n} (t-t_n)^2 f_n'''(t)\,dt\right| \le \frac{\delta_n^3}{2}\sup_{|t-t_n|\le\delta_n} |f_n'''(t)| = \frac{\delta_n^3 t_n^2 e^{-t_n^2/2}}{2\sqrt{2\pi}}(1+o(1)) = \frac{\lambda^3}{1024c_n}(1+o(1)).$$

Substituting from (25), we then have for a small enough choice of λ that

$$f_n(t_n \pm \delta_n) - f_n(t_n) = -\frac{\lambda^2}{128c_n}[1 - O(\lambda)][1+o(1)] \qquad (n\to\infty).$$

Substituting in (31), we have as $n \to \infty$

$$\int_{|t-t_n|\ge\delta_n} e^{nf_n(t)}\,dt \lesssim e^{nf_n(t_n)} \cdot \left\{\sqrt{2\pi c_n}\exp\left(-\frac{\lambda^2 n}{128c_n}[1+o(n)][1-O(\lambda)]+\zeta\right)\right\}. \quad (33)$$

Noting that $\zeta = \sup|f_n(t_n)|$ is finite, and that $n/c_n \sim n/d_n \to \infty$ as $n \to \infty$, (12) follows from (32) and (33) by choosing λ suitably small.

The proof for a choice of margin $\mathcal{B} = \beta p_n c_n^\alpha$ with $0 < \alpha < 1$ and $\beta > 0$ is similar (with (24) giving the asymptotic form for t_n in this case). □

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Convergence of the Calculation of the Innovation Process

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Abstract—**The calculation of the innovation process plays a significant role in the solution of the nonlinear filtering problem. Convergence of an approximate calculation of the innovation process is proved, and then illustrated.**

Index Terms—**Estimation, filtering, semimartingale, Ito stochastic integral, Ito formula, innovation process, infinite expansions.**

I. Introduction

Consider the following partially observed system:

$$\begin{cases} x_t, & \\ y_t = \int_0^t h(s, x_s)ds + w_t, & \end{cases} \quad 0 \le t \le T,$$

where $x = (x_t)_{t\in[0,T]}$ is a measurable d-dimensional real-valued process on a complete probability space $(\Omega, \mathcal{F}, P)$, $w = (w_t)_{t\in[0,T]}$

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is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $h(s,x)$ is a Borel measurable function on $[0,T] \times R^d$. Now let $\mathcal{F}_t^y := \sigma\{y_s : 0 \le s \le t\}$, and similarly for $\mathcal{F}_t^{x,y}$. Assume that $(\mathcal{F}_t^y)_{t\in[0,T]}$, $(\mathcal{F}_t^{x,y})_{t\in[0,T]}$ satisfy the usual conditions, i.e, they are right-continuous and contain all P-null sets in $\mathcal{F}$. The filtering problem consists of seeking a good estimate for the function of the state process x_t, given the observations of y up to a certain time t.

As is well known, no explicit solution to this problem, except for very special cases [6], has been found. So various authors have searched for approximate solutions (see, e.g., [1], [7]). In particular, D.Ocone [7], using multiple integral series expansions, expressed the conditional expectation as a ratio of two multiple integral series, and obtained some finite expansion approximations. However, the coefficients of these finite expansion approximations are not easy to calculate. In [8], we expanded the solution (i.e., the conditional expectation) as an infinite series in terms of the iterated integral with respect to the innovation process, which was used for the first time to solve the filtering problem by Kailath and others [2]–[4]. But then, the question arises of how to compute this innovation process, which itself constitutes a nonlinear filtering problem. This is the purpose of the present correspondence.

Even though the developments in the present correspondence pertain to continuous-time stochastic systems, the results obtained can be shown to apply to discrete-time systems as illustrated by the example in Section IV.

The correspondence is organized as follows: Section II contains preliminary developments. In Section III, an approximation algorithm is proposed and proved for the innovation process. A numerical example is illustrated in Section IV.

II. Preparation

In what follows, the symbol $\int_0^s pq_{s_1}d\nu_{s_1}$ is defined as $p\int_0^s q_{s_1}d\nu_{s_1}$ when p does not depend on s_1 and $\int_0^s q_{s_1 d\nu_{s_1}}$ is well defined in the sense of Ito w.r.t. some semimartingale $(\nu_s, \mathcal{G}_t)$. Note that p is not necessarily adapted to $\mathcal{G}_t$. And throughout the correspondence, the so-called multiple integral is just the iterated integral. Although the results can be proved for a more general case, we assume that $h(s,x)$ is a bounded function for the sake of simplicity. Thus, even through the results in [8] are all true in the one-dimensional case, they obviously hold when x is a d-dimensional process. We cite some of them here for the sake of convenience. Denote by B the set of $\mathcal{F}_t^y$-adapted processes with upper bound M.

1) $\hat{h}_{t,s} = E_0[h_t\bar{\Lambda}_s|\mathcal{F}_s^y]$, where E_0 is the expectation operator under the probability P_0, which is defined by

$$dP_0 = \exp\left\{-\int_0^T h_s\,dw_s - \frac{1}{2}\int_0^T h_s^2\,ds\right\}dP$$

and

$$\bar{\Lambda}_s := \exp\left\{\int_0^s \left(h_l - \hat{h}_l\right)d\nu_l - \frac{1}{2}\int_0^s \left(h_l - \hat{h}_l\right)^2 ds\right\},$$

where $\nu_s := y_s - \int_0^s \hat{h}_l\,dl$ is the innovation process. It is a $\mathcal{F}^y$-Brownian motion under the probability measure P.

2) $\hat{h}_t = Eh_t + \sum_{i=1}^{\infty}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{i-1}} H_i\left(h_t, \hat{h}; s_1, \cdots, s_i\right)d\nu_{s_i}\cdots d\nu_{s_1}$, where

$$H_i(h_t, u; s_1, \cdots, s_i) := E(h_t h_{s_1}\ \cdots\ h_{s_i}) - \sum_{j=1}^{i} u_{s_j} \cdot E\left(h_t h_{s_1}\ \cdots\ h_{s_{j-1}}h_{s_{j+1}}\ \cdots\ h_{s_i}\right)$$

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